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Effective World-Sheet Theory for Non-Abelian Semilocal Strings in $\mathcal{N} = 2$ Supersymmetric QCD

M. Shifman[1], W. Vinci[1] and A. Yung[1,2]

[1] *William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455, USA*

[2] *Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia*

Abstract

We consider non-Abelian semilocal strings (vortices, or vortex-strings) arising in $\mathcal{N} = 2$ supersymmetric U(N) gauge theory with $N_f = N + \tilde{N}$ matter hypermultiplets in the fundamental representation (quarks), and a Fayet–Iliopoulos term ξ. We present, for the first time ever, a systematic *field-theoretic* derivation of the world-sheet theory for such strings, describing dynamics of both, orientational and size zero modes. Our derivation is complete in the limit $(\ln L) \to \infty$ where L is an infrared (IR) regulator in the transverse plane. In this limit the world-sheet theory is obtained exactly. It is presented by a so far unknown $\mathcal{N} = 2$ two-dimensional sigma model, to which we refer as the zn model, with or without twisted masses. Alternative formulations of the zn model are worked out: conventional and extended gauged formulations and a geometric formulation. We compare the exact metric of the zn model with that of the weighted CP($N_f - 1$) model conjectured by Hanany and Tong, through D-branes, as the world-sheet theory for the non-Abelian semilocal strings. The Hanany–Tong set-up has no parallel for the field-theoretic IR parameter and metrics of the weighted CP($N_f - 1$) model and zn model are different. Still their quasiclassical excitation spectra coincide.

Contents

1 Introduction

The exact results obtained in the mid 1990s transformed a class of $\mathcal{N}=2$ supersymmetric gauge theories into powerful benchmark models allowing one to study, to an extent, non-perturbative physics of real QCD [1, 2]. In the last decade we witnessed an enormous progress in the study of supersymmetric solitonic objects in the same type of theories [3–6]. While one usually constructs solitons in a weakly coupled (Higgsed) regime, it is possible to use supersymmetry to infer the role of solitons at strong coupling. For example, Seiberg and Witten proved that confinement in pure $\mathcal{N}=2$ supersymmetric QCD (SQCD) slightly deformed by a mass term of the adjoint field is due a dual Meissner effect (dual superconductivity): the chromoelectric charges are confined by flux tubes which form due to the monopole condensation [1, 2, 7]. This mechanism was anticipated by Nambu, 't Hooft and Mandelstam in the mid 1970s [8–10].

Certainly, the most interesting discovery in this range of questions is the non-Abelian string (also referred to as vortex-string, or just vortex) [11–14], see also [3–5, 15] for a review. It generalizes the long-known Abrikosov–Nielsen–Olesen (ANO) string [16, 17]: internal moduli describing the orientation of the chromomagnetic flux in the non-Abelian group appear on the string world sheet. Thus, the non-Abelian strings are the bridge that connects solitons appearing in the Coulomb phase with those present in the Higgs phase. Moreover, they provide a physical explanation [13, 14] of remarkable correspondences between two-dimensional sigma models and four-dimensional gauge theories observed previously [18–20].

Non-Abelian strings were first discovered in $\mathcal{N}=2$ SQCD with the gauge group U(N) and $N_f = N$ flavors of fundamental matter hypermultiplets (quarks) [11–14]. Internal dynamics of the orientational zero modes of the non-Abelian string supported by this theory is described by two-dimensional $\mathcal{N}=(2,2)$ supersymmetric CP($N-1$) model living on the string world sheet [11–14]. This result was obtained both by a straightforward field-theoretic derivation and D-brane-based arguments, see [3] for a review. More recently, similar non-Abelian strings were constructed and studied in a more general class of theories [21], including SO(N) and Usp(N) gauge theories [22], and models with arbitrary matter content [23].

When one considers theories with a large number of flavors (i.e. $N_f = N + \tilde{N} > N$), the non-Abelian strings one deals with are essentially of the semilocal type: in addition to translational and orientational moduli, they

acquire some moduli related to their physical size. Semilocal strings are interesting because they interpolate between the ANO-type strings (at vanishing size) and sigma-model lumps (at large sizes) [24–28]. Revealing low-energy dynamics of the semilocal strings we are able to understand their role in non-perturbative physics of the bulk theory. For example, arbitrary thickness of the semilocal strings may be responsible for lost confinement [29]. This issue is related to the semilocal string stability, a property which is not ensured by topology. One has to carefully check this stability explicitly [25, 30, 31].

Derivation of the low-energy effective theory on (non-Abelian) semilocal strings was carried out in the past in the framework of string theory, through a D-brane set-up [11, 14]. The effective theory was identified as a particular type of a linear gauged sigma model with an appropriate matter content, namely two-dimensional CP$(N_f - 1)$ with N positive and $\tilde{N}$ negative charges (the so-called weighted CP model). The latter seems to be a natural generalization of the CP$(N - 1)$ model appearing on the world sheet at $N = N_f$ to the case $N_f > N$ [11, 14].

This construction, known as the Kähler quotient, is similar to the well-known Atiyah–Drinfeld–Hitchin–Manin (ADHM) construction for instantons [32]. Unfortunately, contrary to the instanton case, the Kähler quotient construction for strings is unable, in principle, to describe the correct metric on the moduli space. This is the reason why an honest and direct derivation from field theory *per se* is not only desirable, but, in fact, necessary.

This program started in 2006 [29], with further advances ensuing shortly, in [33, 34], by virtue of a more general formalism known as *moduli matrix*. In these two works the problem was addressed in the limit of the large vortex size, in which the differential BPS equations are reducible to an algebraic system.

In this paper, we undertake a new field-theoretic calculation of the low-energy effective action for a single non-Abelian semilocal string, describing dynamics of both, orientational and size zero modes. Our derivation is complete in the limit $(\ln L) \to \infty$ where L is an infrared (IR) regulator in the transverse plane. In physical terms L is implemented through the (s)quark mass difference, $L = |\Delta m|^{-1}$. In this limit the world-sheet theory is obtained exactly. It is presented by a so far unknown $\mathcal{N} = 2$ two-dimensional sigma model, to which we refer as the *zn* model, with or without twisted masses. The bosonic part of the action of the $\mathcal{N} = (2, 2)$ *zn* model (without twisted

masses) is

$$S_{\text{eff}} = \int d^2x \left\{ |\partial_k(z_j n_i)|^2 + \frac{4\pi}{g^2} \Big[|\partial_k n_i|^2 + (n_i^* \partial_k n_i)^2 \Big] \right\}, \tag{1.1}$$

where $i = 1, ..., N$, while $j = 1, ..., \tilde{N}$. The complex fields n_i are subject to the constraint

$$\sum_{i=1}^{N} n_i^* n_i = 1\,.$$

The latter is familiar from the CP$(N-1)$ models. The additional complex fields z_j, descendants of the size moduli, are unconstrained. As we will see later, this novel model has rich dynamics.

Alternative formulations of the zn model are worked out: conventional and extended gauged formulations as well as a geometric formulation. *En route*, we clarify the disagreement between two works mentioned above [29, 31]. We also explicitly calculate, for the first time, corrections to the metric in inverse powers of the vortex size.

Needless to say, the effective action (1.1) collects only terms quadratic in derivatives. As such, it is valid for low-energy excitations. At energies on the world sheet $\sim |\Delta m|$ higher-derivative terms will become important. We will always limit ourselves to the two-derivative terms.

The leading term in the metric contains an infrared divergence

$$\ln L\sqrt{\xi}, \tag{1.2}$$

regularized by an IR cutoff L, where ξ is the Fayet–Iliopoulos (FI) parameter [35]. The logarithmic divergence above is due to long-range tails of the semilocal string which fall off as *powers* of the distance from the string axis (in the perpendicular plane) rather than exponentially. The fact that the size zero modes of the Abelian semilocal strings are logarithmically non-normalizable was noted long ago [36–38]. In the non-Abelian semilocal strings both the size and orientational moduli become logarithmically non-normalizable [29]. One possibility is to replace an infinite-length string by that of a finite length. This will also regulate the spread of the string in the transverse plane [39]. As was mentioned a more convenient and natural IR regularization, which will maintain the BPS nature of the solution, can be provided by a mass difference $\Delta m \neq 0$ of the (s)quark masses [29]. In this

paper we will keep in mind the latter option, using L as an auxiliary parameter at intermediate stages, which, eventually, will be traded for $1/|\Delta m|$, so that (1.2) becomes

$$\ln\frac{\sqrt{\xi}}{|\Delta m|}\,. \tag{1.3}$$

We will always assume that

$$\frac{\sqrt{\xi}}{|\Delta m|}\gg 1\,, \tag{1.4}$$

and, in the second part, the logarithm of the above parameter will be considered to be (arbitrarily) large.

In our derivation we take advantage of the presence of this IR logarithm in the world-sheet theory. We extract the most singular terms in the limit in which the logarithm (1.3) tends to infinity. This allows us to find the exact metric (in the above limit). In this way we arrive at the zn model on the string world sheet. This model is novel; it was not known so far. We start its investigation and uncover interesting features.

Next, we compare the zn model with the weighted CP(N_f-1) model suggested by Hanany and Tong [11,14] as the world-sheet theory. First, we explicitly verify that our field-theory result is different from the string theory prediction: the scalar curvatures for the two metrics (ours and the Hanany–Tong one) are not the same. Still quasiclassical excitation spectra of two models coincide.

Summarizing, our main task with regards to non-Abelian semilocal strings is two-fold. First, at large ρ (where ρ is the size of the string) we derive the Kähler potential on the target space as an expansion in the powers of $1/|\rho|$, keeping the leading and the first subleading terms. The limit of large L is not used here. The second task, the central point of our paper, is to use the limit $\ln\frac{\sqrt{\xi}}{|\Delta m|}\gg 1$ to derive the *exact* world-sheet model (which, in this case, corresponds to small $\rho\ll\xi^{-1/2}$).

The organization of the paper is as follows. In Sec. 2 we introduce the bulk model, construct the semilocal string and calculate its world-sheet effective action. We derive the corresponding Kähler potential, including the first correction in the inverse size of the string. In Sec. 3 we calculate the exact metric of the world-sheet theory in the limit of the large IR logarithm. Section 4 is devoted to nonvanishing masses introduced in and their impact

on the string world-sheet theory. In Sec. 5 we calculate the quasiclassical spectrum of excitations in the world-sheet theory. In Sec. 6 we review the Hanany–Tong world-sheet theory obtained from D-branes and compare it to our field-theory result. The D-brane derivation is blind to infrared logarithms implying a model different from the zn model. In Sec. 7 we present the world-sheet effective theory below the crossover (i.e. at small ξ) and compare it with the weighted CP(N_f-1) model. Finally, we conclude and summarize our results in Sec. 8.

2 Non-Abelian Semilocal Strings from Field Theory

In this section we will derive the string world-sheet theory in the limit of large ρ, where ρ is a size modulus, $|\rho|^2\xi \gg 1$. In this limit a natural expansion parameter appears, namely the one given in Eq. (2.30) below. We will use it in calculating the effective action to the leading and the first subleading order. Later on (in Sec. 3) we will relax the constraint $|\rho|^2\xi \gg 1$. At first we must introduce our basic bulk model, on which will rely not only in this section, but throughout the paper.

2.1 The Bulk Theory

Our starting point is a U(N) gauge theory with extended $\mathcal{N}=2$ supersymmetry and $N_f = N + \tilde{N}$ fundamental hypermultiplets. The bosonic part of the model[1] is (see e. g. [3])

$$
\begin{aligned}
S &= \int d^4x \left\{ \frac{1}{4g^2}(F^0_{\mu\nu})^2 + \frac{1}{4g^2}(F^a_{\mu\nu})^2 + \frac{1}{g^2}|\partial_\mu \phi^0|^2 + \frac{1}{g^2}|D_\mu \phi^a|^2 + |\nabla_\mu q^A|^2 + \right. \\
&+ \frac{g^2}{2}\left(\frac{1}{g^2} f^{abc}\bar{\phi}^b\phi^c + \bar{q}^A T^a q^A\right)^2 + \frac{g^2}{8}(|q^A|^2 - N\xi)^2 + \\
&+ \frac{1}{2}\left|(\phi^0 \frac{2}{\sqrt{2N}} + \phi^a 2T^a + \sqrt{2}m_A)q^A\right|^2 ,
\end{aligned}
\tag{2.1}
$$

[1]The complete bosonic sector includes, in addition, N_f antifundamental multiplets $\tilde{q}^A$. We set them to zero, $\tilde{q}^A = 0$, as they are trivial in the classical configurations to be discussed below.

with:

$$A = 1, 2, \dots, N_f \qquad \nabla_\mu = \partial_\mu - \frac{i}{\sqrt{2N}} A^0_\mu - i T^a A^a_\mu \,. \tag{2.2}$$

The real parameter ξ is the Fayet–Iliopoulos (FI) term [35]. As we will see shortly, a nonvanishing ξ puts the theory into the Higgs phase. Moreover, the superscripts 0 and a refer to the U(1) and SU(N) parts of the gauge group, respectively. For simplicity we choose both gauge couplings to be equal. This assumption is not necessary and could have been readily lifted, but we prefer to work with a single gauge coupling g. If the mass parameters m_A are taken real, we can consistently consider the adjoint fields a^0, a^a to be real as well on the solitonic solutions. The above expression then simplifies,

$$\begin{aligned} S \;&=\; \int d^4x \left\{ \frac{1}{4g^2}(F^0_{\mu\nu})^2 + \frac{1}{4g^2}(F^a_{\mu\nu})^2 + \frac{1}{g^2}|\partial_\mu \phi^0|^2 + \frac{1}{g^2}|D_\mu \phi^a|^2 + |\nabla_\mu q^A|^2 + \right. \\ &+ \left. \frac{g^2}{2}\left(\bar{q}^A T^a q^A\right)^2 + \frac{g^2}{8}(\bar{q}^A q^A - N\xi)^2 + \frac{1}{2}\left| \left(\phi^0 \frac{2}{\sqrt{2N}} + \phi^a 2T^a + \sqrt{2} m_A \right) q^A \right| \right\} . \end{aligned} \tag{2.3}$$

It is convenient to organize all fields into matrices, of sizes $N \times N$ and $N \times N_f$, respectively,

$$F_{\mu\nu} \equiv F^0_{\mu\nu} \frac{\mathbf{1}_N}{\sqrt{2N}} + F^a_{\mu\nu} T^a, \quad \Phi \equiv \sqrt{2}\left(\phi^0 \frac{\mathbf{1}_N}{\sqrt{2N}} + \phi^a T^a \right), \quad Q \equiv q^A_i \,. \tag{2.4}$$

Using the notation above, the action (2.3) can be written in the following compact form:

$$S = \int d^4x \,\mathrm{Tr}\, \left\{ \frac{1}{2g^2} F^2_{\mu\nu} + \frac{1}{g^2}|D_\mu \Phi|^2 + |\nabla_\mu Q|^2 + \frac{g^2}{4}(Q\bar{Q} - \xi)^2 + |\Phi Q + Q M|^2 \right\} , \tag{2.5}$$

where the square mass matrix M is defined as

$$M_{AB} = \delta_{AB} m_A = \begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & m_{N_f} \end{pmatrix} . \tag{2.6}$$

The nonvanishing (s)quark masses break the $SU(N)_{\rm F}$ flavor symmetry down to $U(1)_{\rm F}^{N-1}$. Note that we can always absorb a unit mass matrix into a shift of the adjoint field Φ. Then, with no loss of generality, we can always set

$$\sum_{A=1}^{N} m_A = 0\,.$$

This model described in great detail in [3] has a number of isolated vacua at generic masses. We choose the vacuum where first N quark flavors condense. Up to gauge symmetry transformations we have

$$\Phi_0 = -M, \quad Q = \sqrt{\xi}\ (1_N, 0_{\tilde{N}})\,. \tag{2.7}$$

This vacuum is invariant under a "color-flavor locked" global symmetry $H_{\rm C+F}$,[2]

$$H_{\rm C+F}(\Phi) = H_{\rm C}\Phi H_{\rm C}^{-1} = \Phi\,, \quad H_{\rm C+F}(Q) \equiv H_{\rm C}\, Q\, H_{\rm F}^{-1}, \quad H_{\rm C} = H_{\rm F}\,. \tag{2.8}$$

The above symmetry plays an important role in the study of the moduli space of solitons. It is determined by the vacuum value of Φ. In the most general case, in which some of the mass parameters are degenerate, it is given by the stabilizer of the adjoint field,

$$H_{\rm C+F} = {\rm S}({\rm U}(n_1)\times {\rm U}(n_2)\cdots \times {\rm U}(n_q)), \quad n_1+\cdots+n_q = N\,. \tag{2.9}$$

Equation (2.9) assumes that there are q sets of fields with degenerate masses. The theory has two parameters with mass dimension one, $m \sim m_i$ and $\sqrt{\xi}$, while the dynamical scale Λ is implicit.[3] For the time being we will impose the constraints

$$m \ll \sqrt{\xi}\,, \qquad \Lambda \ll \sqrt{\xi}\,. \tag{2.10}$$

Then the bulk theory is at weak coupling, and we can reliably deal with the (s)quark masses as small deformations of the world-sheet theory. In this regime, the symmetry breaking pattern reduces to

$${\rm U}(N)_C \times {\rm SU}(N)_F \xrightarrow{\sqrt{\xi}} {\rm SU}(N)_{C+F} \xrightarrow{m} H_{C+F}. \tag{2.11}$$

[2]Note that the vacuum is also invariant under an additional $H_F = {\rm S}({\rm U}(\tilde{n}_1)\times\cdots\times{\rm U}(\tilde{n}_p))$ flavor symmetry ($\tilde{n}_1+\cdots+\tilde{n}_p = \tilde{N}$).

[3]For convenience we chose the masses m_i to be all of the same order m.

In particular, in the equal-mass limit the global symmetry group of the bulk theory is

$$\mathrm{SU}(N)_{C+F} \times \mathrm{SU}(\tilde{N})_F \times \mathrm{U}(1)\,, \tag{2.12}$$

broken down to $\mathrm{U}(1)^{(N_f-1)}$ by generic quark mass differences.

At the quantum level, the theory develops a strong coupling scale Λ. The Higgsing at the scale $\sqrt{\xi}$ freezes the one-loop running of the gauge coupling at the value

$$\frac{8\pi^2}{g^2} = (N - \tilde{N}) \ln \frac{\sqrt{\xi}}{\Lambda}\,. \tag{2.13}$$

The theory is asymptotically free for $N > \tilde{N}$, and conformally invariant at $N = \tilde{N}$. We will assume, in the following $N > \tilde{N}$. For large values of the FI term, $\xi \gg \Lambda$, weak coupling regime sets in (complete Higgsing!), and we can reliably construct semiclassical vortex solutions.

BPS equations

The first step in the studies of the BPS-saturated strings is to consider the set of the first-order differential equations known as the Bogomol'nyi equations [40], which follow from the Bogomol'nyi completion of the action (2.5) [4,11–14,18,41],

$$\begin{aligned} S &= \int d^4x\, \mathrm{Tr} \left\{ \frac{1}{g^2} \left(F_{12} + \frac{g^2}{2}(Q\bar{Q} - \xi) \right)^2 + \right. \\ &+ \ |\nabla_1 Q + i \nabla_2 Q|^2 + |\Phi Q + QM|^2 + \xi\, F_{12} + \\ &+ \left. \frac{1}{g^2}(F_{ik})^2 + (\nabla_k Q)^* (\nabla_k Q) + \frac{1}{g^2}(F_{kl})^2 \right\}, \\ & \quad i = 1,2, \quad k,l = 0,3\,. \end{aligned} \tag{2.14}$$

In our notation, $i = 1,2$ denotes the transverse (with respect to the string) directions, while $k = 0,3$ are the space-time coordinates on the string world-sheet.

The Bogomol'nyi equations are obtained, for static solutions, by requiring each positive-definite contributions above to vanish,

$$\nabla_1 Q + i\nabla_2 Q = 0\,,$$
$$F_{12} + \frac{g^2}{2}(Q\bar{Q} - \xi) = 0\,. \tag{2.15}$$

The string tension is given by the last term in the second line in (2.14), the topological term,

$$T = \xi \int d^2x \,\mathrm{Tr}\, F_{12} = 2\pi\xi\, n\,, \tag{2.16}$$

where n is the quantized magnetic flux, or equivalently, the total number of strings.

2.2 Non-Abelian Semilocal Strings: $\tilde{N} = 1$

In this section we will consider the simplest theory which supports semilocal strings, with a single "additional" flavor, $\tilde{N} = 1$. Semilocal strings are present when the set of vacua of the theory is not simply-connected [27]. Actually, the correct topological object to examine in connection with the semilocal strings is the second homotopy group of the vacuum manifold, which, in the present case, is the complex projective space,

$$\pi_2(\mathcal{M}_{\mathrm{vac}}) = \pi_2\left(\mathrm{CP}(N-1)\right) = \mathbb{Z}\,. \tag{2.17}$$

Equation (2.17) is relevant for the extension of the ANO string in the corresponding semilocal string. The homotopy group in (2.17) is the one lying behind the description of lumps in the associated nonlinear sigma-model, which arises as the low-energy limit of the theory (2.1). This is the main reason why semilocal strings are similar to lumps [26, 33, 34]. As lumps, the semilocal strings have power-law behaviors at large distances, and possess new *size* moduli determining their characteristic thickness. Nevertheless, they still retain their nature of strings (flux tubes), which is manifest when we send the size moduli to zero. In this limit we recover just the ANO string, with its exponential behavior [24].

Topological stability of the non-Abelian strings is due to the fact that

$$\mathbb{Z}_N \in \mathrm{U}(1) \;\text{ and }\; \pi_1(\mathrm{U}(1)\times \mathrm{SU}(N)/\mathbb{Z}_N) = \mathbb{Z}\,. \tag{2.18}$$

Much in the same way as the ANO string, they can be elevated to the semilocal strings, see [3]. The winding structure inherent to the non-Abelian vortices, in the context of the semilocal strings, is discussed in the subsequent sections, Eq. (2.19) and below.

2.2.1 Ansatz

Our task is to explicitly construct a single semilocal string. For the time being we will set all mass parameters to zero. This is the situation when the full color-flavor symmetry $\mathrm{SU(N)}_{C+F}$ is preserved, and strings develop size moduli.[4] As was shown in [11–14], we can easily embed the Abelian ANO-type string into a larger non-Abelian gauge group to obtain the so-called $\mathbb{Z}_N$ string [3]. This can be done by choosing the following ansatz for the matter fields [29]:

$$Q_0 = \left(\begin{array}{cccc|c} \phi_1(r) & 0 & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \dots & \phi_1(r) & 0 & 0 \\ 0 & \dots & 0 & \phi_2(r) e^{i\theta} & \phi_3(r) \end{array} \right) . \tag{2.19}$$

Equation (2.19) corresponds to a special embedding in which a nontrivial topological winding is provided by the N-th flavor. Technically, it is more convenient to work in the singular gauge in which the fields assume the following form:

$$\begin{aligned} Q_0 &= \left(\begin{array}{ccc|c} \phi_1(r) & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \phi_2(r) & \phi_3(r) \end{array} \right) \equiv \\ &\equiv \Big(\phi_1(r) - n_0\, n_0^* (\phi_1(r) - \phi_2(r)) \,|\, n_0\, \phi_3(r) \Big) , \end{aligned} \tag{2.20}$$

while the gauge fields are

$$A_{0,i} = \epsilon_{ij} \frac{x_j}{r^2} f(r) \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix} \equiv n_0\, n_0^*\, \epsilon_{ij} \frac{x_j}{r^2} f(r) . \tag{2.21}$$

[4]We will reintroduce masses in Sec. 4

In the expressions above we introduced an N-component vector n_0 transforming in the fundamental representation of the color flavor group $H_{\mathrm{C+F}}$,

$$n_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} , \quad n \equiv H_{\mathrm{C+F}}\, n_0 \,. \tag{2.22}$$

Given the $\mathbb{Z}_N$ string, and acting on the solution (2.21) with a generic color-flavor transformation, we get the most general vortex-string solution in terms of the orientational vector n,

$$\begin{aligned} Q &= \Big(\phi_1(r) - n\, n^*(\phi_1(r) - \phi_2(r)) \,|\, n\, \phi_3(r)\Big) \,, \\ A_i &= n\, n^*\, \epsilon_{ij} \frac{x_j}{r^2} f(r) \,, \end{aligned} \tag{2.23}$$

where the complex N-vector n_i is obviously subject to the condition

$$|n_i|^2 = 1. \tag{2.24}$$

2.2.2 BPS equations and solutions

The non-Abelian Bogomol'nyi equations reduce to those of the Abelian extended Higgs model. With the ansatz (2.23), we get the following set of equations:

$$\begin{aligned} & r\phi_1'(r) = 0 \,, \\ & r\phi_2'(r) - f(r)\, \phi_2(r) = 0 \,, \\ & r\phi_3'(r) - (f(r) - 1)\phi_3(r) = 0 \,, \\ & \frac{1}{r} f'(r) + \frac{g^2}{2}(\phi_2^2(r) + |\phi_3(r)|^2 - \xi) = 0 \,. \end{aligned} \tag{2.25}$$

Note that the first and third equations for ϕ_1 and ϕ_3 can be identically solved by

$$\phi_1(r) = \sqrt{\xi}, \quad \phi_3 = \frac{\rho}{r}\, \phi_2 \,. \tag{2.26}$$

In the expression above, ρ is a complex modulus which parametrizes the size of the semilocal string. The remaining set of two coupled differential equations, then, must be solved numerically, since no analytical solution is known to exist.

Nevertheless, the peculiarity of the system above is that it admits regular and smooth solutions in the limit of large gauge coupling (the so-called sigma model limit), $g \to \infty$. It is even more remarkable that the same system can be solved algebraically at any finite power in a $1/g^2$ expansion. Keeping only the terms of the order of $1/g^2$, the solution is

$$\phi_2 = \phi_{2,0} + \frac{1}{g^2}\delta\phi_2 \quad = \quad \sqrt{\xi}\frac{r}{\sqrt{r^2+|\rho|^2}} + \frac{1}{g^2}\delta\phi_2\,,$$

$$f = f_0 + \frac{1}{g^2}\delta f \quad = \quad \frac{|\rho^2|}{r^2+|\rho|^2} + \frac{1}{g^2}\delta f\,,$$

$$\delta\phi_2 = -\frac{1}{\sqrt{\xi}}\frac{2r|\rho|^2}{(r^2+|\rho|^2)^{5/2}}\,, \qquad \delta f = \frac{8}{\xi}\frac{r^2|\rho|^2}{(r^2+|\rho|^2)^3}\,. \tag{2.27}$$

If we analyze more carefully the validity of the power expansion, by imposing the conditions

$$\delta\phi_2/(g^2\phi_{2,0}) \ll 1\,, \quad \delta f/(g^2 f_0) \ll 1\,, \tag{2.28}$$

we find

$$\frac{1}{g\sqrt{\xi}|\rho|} = \frac{\lambda_{\rm loc}}{\lambda_{\rm semi}} \ll 1\,, \qquad \lambda_{\rm loc} = \frac{1}{g\sqrt{\xi}}, \quad \lambda_{\rm semi} = |\rho|\,. \tag{2.29}$$

Thus, the correct expansion parameter is

$$1/(g\sqrt{\xi}|\rho|)\,, \tag{2.30}$$

or the ratio of the semilocal string size to the characteristic size of the local string.

2.2.3 The Effective Action

To calculate the effective action on the string world sheet, one first must promote the orientational and size moduli to fields depending on the world-sheet coordinates t and $z \equiv x_3$,

$$n \to n(t, x_3), \quad \rho \to \rho(t, x_3)\,. \tag{2.31}$$

In doing so one has to improve the ansatz (2.23) by including a nontrivial expression for the world-sheet components of the gauge potential [13], namely,

$$\begin{aligned} A_k &= -i\Big(\partial_k n\, n^* - n\,\partial_k n^* - 2n\, n^*(n^*\partial_k n)\Big)\omega(r) \\ &\quad -i\, n\, n^*\Big(\rho^*\partial_k\rho - \rho\partial_k\rho^* + 2|\rho|^2(n^*\partial_k n)\Big)\gamma(r)\,, \end{aligned} \tag{2.32}$$

where we introduced two profile functions $\omega(r)$ and $\gamma(r)$, to be determined from a minimization procedure. Note that expression (2.32) is a refined ansatz as compared to the one introduced in Refs. [13, 29], which does not includes the second term proportional to γ. The resulting expression for the field strength is

$$\begin{aligned} F_{ik} &= \partial_i A_k - \partial_k A_i - i[A_i, A_k] \\ &= -\partial_k(n\, n^*)\epsilon_{ij}\frac{x_j}{r^2}f(r)(1-\omega(r)) \\ &- i\Big(\partial_k n\, n^* - n\,\partial_k n^* - 2n\, n^*(n^*\partial_k n)\Big)\frac{x_i}{r}\omega'(r) \\ &- i\, n\, n^*\Big(\rho^*\partial_k\rho - \rho\partial_k\rho^* + 2|\rho|^2(n^*\partial_k n)\Big)\frac{x_i}{r}\gamma'(r) \\ &- n\, n^*\epsilon_{ij}\frac{x_j}{r^2}\partial_k f(r)\,, \end{aligned} \tag{2.33}$$

where the prime in ω' and γ' stands for the first derivative with respect to r.

As a second step, we evaluate the action (2.14) on the semilocal solution (2.23), in conjunction with (2.32). We will keep the terms quadratic in the time derivatives with respect to the world-sheet coordinates,

$$\mathcal{L}_{\rm eff} = \int dx_1 dx_2\, \mathrm{Tr}\left\{\frac{1}{g^2}(F_{ik})^2 + (\nabla_k Q)^*(\nabla_k Q)\right\}\,. \tag{2.34}$$

2.2.4 The Gauge Kinetic Term

Evaluation of the gauge kinetic term using the above ansätze is straightforward,

$$\begin{aligned}
\frac{1}{g^2}\mathrm{Tr}\,(F_{ik})^2 &= \frac{1}{g^2}\left(\frac{2}{r^2}f^2(1-\omega)^2+2\omega'^2\right)\left[\partial_k n^*\partial_k n+(\partial_k n^* n)^2\right] \\
&- \frac{1}{g^2}\left(\gamma'^2\right)\left[\rho^*\partial_k\rho-\rho\partial_k\rho^*+2|\rho|^2(n^*\partial_k n)\right]^2 \\
&+ \frac{1}{g^2}\frac{1}{r^2}(\partial_{|\rho|^2}f)^2\left[\partial_k|\rho|^2\right]^2\,. \qquad (2.35)
\end{aligned}$$

2.2.5 The Matter Fields

Now we pass to the matter fields. Referring the reader to Appendix A for details we present here the result of a straightforward albeit rather tedious calculation,

$$\begin{aligned}
&\mathrm{Tr}\left[(\nabla_k Q)^*(\nabla_k Q)\right] \\
=\;& \left[2(\sqrt{\xi}-\phi_2)^2(1-\omega)+\frac{|\rho|^2}{r^2}|\phi_2|^2(1-2\omega)+\left(\xi+|\phi_2^2|(1+\frac{|\rho|^2}{r^2})\right)\omega^2\right] \\
\times\;& \left[\partial_k n^*\partial_k n+(\partial_k n^* n)^2\right] \\
+\;& \left[\left(1+\frac{|\rho|^2}{r^2}\right)(\partial_{|\rho|^2}\phi_2)^2+\frac{1}{r^2}\phi_2\,\partial_{|\rho|^2}\phi_2\right](\partial_k|\rho|^2)^2 \\
+\;& \frac{1}{|r|^2}|\phi_2|^2|\partial_k\rho+\rho(n^*\partial_k n)|^2+\frac{1}{r^2}|\phi_2|^2(\rho^*\partial_k\rho-\rho\partial_k\rho^*+2|\rho|^2\,n^*\partial_k n)^2\gamma \\
-\;& |\phi_2^2|(1+\frac{|\rho|^2}{r^2})(\rho^*\partial_k\rho-\rho\partial_k\rho^*+2|\rho|^2\,n^*\partial_k n)^2\gamma^2\Bigg\}\,, \qquad (2.36)
\end{aligned}$$

where we took advantage of the exact (to all orders in $1/g^2$) equations (2.26).

2.2.6 Determination of $\omega(r)$ and $\gamma(r)$

To determine the profile functions $\omega(r)$ and $\gamma(r)$, we have to minimize the expression given by the sum of two pieces (2.35) and (2.36). The minimization

with respect of $\omega(r)$ was performed in Refs. [13, 29]. It gives the following differential equation:

$$
\begin{aligned}
- & \quad \frac{2}{g^2}\omega'' - \frac{2}{g^2 r}\omega' - \frac{2}{g^2 r^2} f^2(1-\omega) + \big(\xi + \phi_2^2 + \frac{|\rho|^2}{r^2}\phi_2^2\big)\omega \\
- & \quad (\xi - \phi_2)^2 + \frac{|\rho|^2}{r^2}\phi_2^2 = 0\,, \qquad (2.37)
\end{aligned}
$$

which is exactly solved by

$$
\omega = 1 - \frac{\phi_2}{\sqrt{\xi}}\,. \qquad (2.38)
$$

Minimization with respect to γ gives, on the other hand,

$$
\tfrac{2}{g^2}\gamma'' + \tfrac{2}{g^2 r}\gamma' + \tfrac{1}{r}\phi_2^2 - 2r\phi_2^2\big(1 + \tfrac{|\rho|^2}{r^2}\big)\gamma = 0\,. \qquad (2.39)
$$

The equation above is solved algebraically at zeroth order in $1/g^2$ by

$$
\gamma = \frac{1}{2}\frac{1}{r^2 + |\rho|^2} + \frac{1}{g^2}\delta\gamma\,.
$$

We do not evaluate explicitly the term $\delta\gamma$, since it turns out that it does not contribute, at the same order $1/g^2$, to the effective action.

2.2.7 $1/(g^2\xi|\rho|^2)$ corrections to the effective action

We now have all ingredients necessary to calculate the low-energy effective action for the non-Abelian semilocal string, up to the order $1/(g^2\xi|\rho|^2)$. By evaluating the action given by (2.35) and (2.36), exploiting the expressions for the fields (2.27) and (2.38), and integrating over the transverse plane, we arrive at [5]

$$
\begin{aligned}
\mathcal{L}_{\rm eff} &= \pi\xi\left(\ln\frac{L^2}{|\rho|^2}\right)\left|\partial_k(\rho\, n)\right|^2 - \pi\xi|\partial_k\rho + \rho\,(n^*\partial_k n)|^2 \\
&+ \frac{2\pi}{g^2}\left[\partial_k n^*\partial_k n + (\partial_k n^* n)^2\right]. \qquad (2.40)
\end{aligned}
$$

[5] See Appendix A for more details.

The first term explicitly exhibits the infrared divergence mentioned in Sec. 1. An IR divergent integral in the perpendicular plane is cut off at L at large distances. Thus, the large-size constant L is introduced to keep the integrations over the transverse plane finite. This divergent term was first calculated in Ref. [29]. In this paper we used the very same approach in order to obtain its correct expression, which is now consistent with the results of Refs. [33, 34], obtained through the moduli matrix formalism. The last term in Eq. (2.40) is a finite contribution to the metric corresponding to the standard Fubini–Study metric on CP$(N-1)$.

2.2.8 The Kähler Potential

The effective action (2.40) describes 1/2-BPS saturated solitons preserving $\mathcal{N}=(2,2)$ supersymmetry in 1+1 dimensions, on the world sheet. As such, the metric of the world-sheet sigma model must be given by a Kähler potential,

$$g_{\phi_l,\phi_{\bar{m}}} = \partial_{\phi_l}\partial_{\phi_{\bar{m}}} K(\phi_l,\phi_{\bar{m}})\,. \tag{2.41}$$

Now we will establish its form.

To begin with, let us first introduce a set of holomorphic coordinates b_i and c on the target space,

$$b_i = \frac{n_i}{n_N}, \quad c = \rho\, n_N\,, \qquad i = 1, ..., N-1\,, \tag{2.42}$$

implying that

$$|\rho|^2 = (1+\sum_i |b_i|^2)|c|^2\,. \tag{2.43}$$

It is not difficult to show, after some algebra, that the following Kähler

potential gives the correct metric on the target space:

$$
\begin{aligned}
K_{\text{eff}}(b_i, c, \bar{b}_i, \bar{c}) &= \pi\xi \left(1 + \sum_i |b_i|^2\right) |c|^2 \;\ln \frac{L^2}{(1 + \sum_i |b_i|^2)|c|^2} \\
&+ \pi\xi(1 + \sum_i |b_i|^2)|c|^2 + \frac{2\pi}{g^2} \ln(1 + \sum_i |b_i|^2) \\
&= \pi\xi\, |\rho|^2 \left(\ln \frac{L^2}{|\rho|^2}\right) + \pi\xi|\rho|^2 + \frac{2\pi}{g^2} \ln \left(1 + \sum_i |b_i|^2\right) .
\end{aligned}
\tag{2.44}
$$

Note that the above expression is invariant under the color-flavor isometry, as it should be, of course. The first two terms depend only on the physical size $|\rho|$, which is an obvious invariant. The last term, on the other hand, is invariant up to Kähler transformations.

Let us recall here that the Kähler potential in the case of the *local* non-Abelian string, when $\rho = 0$ takes the form [11–14, 41]

$$
K_{\text{eff}}(b, 0, \bar{b}, 0) = \frac{4\pi}{g^2} \ln \left(1 + \sum_i |b_i|^2\right) . \tag{2.45}
$$

We would like to draw attention to the difference of a factor two in the coefficients in front of the logarithms in the expressions (2.44) and (2.45). These two terms do not have to coincide, since they are calculated in the opposite limits $\rho \to \infty$ and $\rho \to 0$, respectively .

2.3 $\tilde{N} > 1$

Now we will lift the requirement $\tilde{N} = 1$. Generalization to a generic number of flavors requires more algebra, but is quite straightforward. One has to introduce $\tilde{N}$ complex size moduli ρ_j, one for for each additional flavor. $\tilde{N}$ BPS equations for the fields q^{N+j} are now exactly solved by the following ansatz:

$$
q^{N+j} = \frac{\rho_j}{r}\, n\, \phi_2(r), \qquad j = 1, \ldots, \tilde{N} . \tag{2.46}
$$

The ansatz for the gauge potential must be also generalized, namely,

$$\begin{aligned} A_k &= -i\Big(\partial_k n\, n^* - n\,\partial_k n^* - 2n\, n^*(n^*\partial_k n)\Big)\omega(r) \\ &\quad -i\, n\, n^* \sum_j^{\tilde N}\Big(\rho_j^*\partial_k\rho_j - \rho_j\partial_k\rho_j^* + 2\rho_j\rho_j^\dagger(n^*\partial_k n)\Big)\gamma(r)\,. \end{aligned} \tag{2.47}$$

The total size of the string $|\rho|^2$ is now given by

$$|\rho|^2 = \sum_{j=1}^{\tilde N} |\rho_j|^2\,. \tag{2.48}$$

Taking into account these relatively insignificant modifications, one gets the effective action in the form

$$\begin{aligned} \mathcal{L}_{\rm eff} &= \pi\xi\left(\ln\frac{L^2}{|\rho|^2}\right)\sum_j |\partial_k(n\rho_j)|^2 - \pi\xi\frac{1}{|\rho|^2}\Big|\sum_j(\rho_j^* n^*)\partial_k(n\rho_j)\Big|^2 \\ &- \frac{2\pi}{g^2}\left(\frac{1}{|\rho|^2}\sum_j|\partial_k(n\rho_j)|^2 - \frac{1}{|\rho|^4}\Big|\sum_j(\rho_j^* n^*)\partial_k(n\rho_j)\Big|^2\right) \\ &+ \frac{4\pi}{g^2}\Big[\partial_k n^*\partial_k n + (\partial_k n^* n)^2\Big]\,. \end{aligned} \tag{2.49}$$

The Kähler potential

Again, it is convenient to introduce a Kähler potential for (2.49). We introduce a set of holomorphic coordinates b_i and c_j, in parallel with (2.42),

$$\begin{aligned} b_i &= \frac{n_i}{n_N}, \qquad c_j = \rho_j\, n_N\,, \\ |\rho|^2 &= \Big(1+\sum_i |b_i|^2\Big)\sum_j |c_j|^2\,, \\ i &= 1,...,N-1, \qquad j = 1,\ldots,\tilde N\,. \end{aligned} \tag{2.50}$$

Then, in terms of these coordinates we have

$$K_{\rm eff}(b_i, c_j, \bar b_i, \bar c_j) = \pi\xi\,|\rho|^2 \ln\frac{L^2}{|\rho|^2} + \pi\xi|\rho|^2 - \frac{2\pi}{g^2}\ln(|\rho|^2)$$

$$+ \quad \frac{4\pi}{g^2}\ln\left(1+\sum_i |b_i|^2\right)$$

$$= \quad \pi\xi\,|\rho|^2 \ln\frac{L^2}{|\rho|^2} + \pi\xi|\rho|^2 - \frac{2\pi}{g^2}\ln\left(\sum_j |c_j|^2\right) + \frac{2\pi}{g^2}\ln\left(1+\sum_i |b_i|^2\right). \tag{2.51}$$

3 Exact World-Sheet Theory

In this section we take advantage of the presence of the IR divergent term $\ln L/|\rho| \gg 1$ in the Kähler potential. Considering the infrared logarithm as a large parameter (in fact, the largest) we relax the condition $\rho g\sqrt{\xi} \gg 1$ and obtain the *exact* metric of the world-sheet theory to the leading order in the IR logarithm. By saying 'exact' we mean that there is no expansion in $1/(|\rho|^2\,\xi)$ in this metric, unlike the results in Sec. 2.2.8 or 2.3.

Consider first the case $\tilde N = 1$. The world-sheet theory (2.40) has the form

$$S_{\rm eff} = \int d^2x \left\{ 2\pi\xi\,\ln\frac{L}{|\rho|}\,|\partial_k(\rho\, n_i)|^2 + \text{finite terms} \right\}, \tag{3.1}$$

where finite terms are those which do not contain the infrared logarithm. We stress that our derivation in Sec. 2 gives us the exact expression in front of the IR logarithm. The reason is that the logarithmically divergent term in the world-sheet theory comes from the long-range power tails of the string solution which we know exactly. Corrections to the string solution associated with the string core at $r \sim 1/g\sqrt{\xi}$ (which we do not control) do not produce the infrared divergent terms in the world-sheet action.

Following [29] we introduce a new variable z replacing the ρ modulus

$$z = \rho \left[2\pi\xi \, \ln \frac{L}{|\rho|} \right]^{1/2} . \tag{3.2}$$

With the *logarithmic accuracy* we rewrite the world-sheet theory in terms of this new variable z as

$$S_{\text{eff}} = \int d^2x \left\{ |\partial_k (z n_i)|^2 + \text{finite terms} \right\} , \tag{3.3}$$

where in the finite terms we have to express ρ in terms of z. We will justify momentarily that in the path integral the field $z(x)$ is not large. In fact it is of order of one, $z \sim 1$. Given that the IR logarithm is large this means that ρ is in fact small. This means that we can take the limit $\rho \to 0$ in the finite terms in Eqs. (3.1) and (3.3). With vanishing ρ, the semilocal non-Abelian string reduces to the usual (local) non-Abelian string, for which world sheet theory is given by the CP$(N-1)$ model [11–14]. Thus, we can write

$$\text{finite terms}\,|_{\rho\to 0} \to \text{CP}(N-1) \text{ model} , \tag{3.4}$$

and the bosonic part of the action of the world-sheet theory takes the form

$$S_{\text{eff}} = \int d^2x \left\{ |\partial_k (z n_i)|^2 + \frac{4\pi}{g^2} \left[|\partial_k n_i|^2 + (n_i^* \partial_k n_i)^2 \right] \right\} . \tag{3.5}$$

In terms of new variables the infrared logarithm disappeared! Now it is clear that typical fluctuations of the z field are $z \sim 1$. In Sec. 4 we will introduce mass terms in (3.5) which will make this observation even more evident.

Equation (3.5) presenting a new world-sheet model in the semilocal string problem, to replace that of Hanany and Tong, is one of our main results.

We stress that the only approximation used here is that the infrared logarithm is large,

$$\ln \, (Lg\sqrt{\xi}) \gg 1. \tag{3.6}$$

In fact, in order to write (3.4) and (3.5) we need ρ to be much smaller than the string core, $\rho \ll 1/g\sqrt{\xi}$. In terms of the field z this reduces to

$$|z|^2 \ll \frac{1}{g^2} \, \ln \, (L \, g\sqrt{\xi}), \tag{3.7}$$

which is obviously satisfied in the limit $L \to \infty$.

In Sec. 4 we will introduce the (s)quark mass terms and show that in this case the infrared cutoff L is replaced by the inverse of a typical mass difference $L \to 1/\Delta m$. Then, instead of (3.7) we have

$$|z|^2 \ll \frac{1}{g^2} \ln\left(\frac{g\sqrt{\xi}}{\Delta m}\right). \tag{3.8}$$

The latter condition (3.7) is still satisfied provided Δm is taken small enough. The parameter $g\sqrt{\xi}$ determines the size of the string core and should be understood as an ultraviolet (UV) cutoff for the low-energy effective world-sheet theory (3.5), see for example [3].

Now, let us generalize (3.5) to the case $\tilde{N} > 1$. Starting with (2.49) and following the same steps which leads us to (3.5) we get

$$S_{\rm eff} = \int d^2x \left\{ |\partial_k(z_j n_i)|^2 + \frac{4\pi}{g^2}\left[|\partial_k n_i|^2 + (n_i^* \partial_k n_i)^2 \right] \right\}, \tag{3.9}$$

where $i = 1, ..., N$, while $j = 1, ..., \tilde{N}$ and we introduced new fields z_j,

$$z_j = \rho_j \left[2\pi\xi \, \ln\frac{L}{|\rho|} \right]^{1/2}. \tag{3.10}$$

Eq. (3.9) is our final result for the effective low energy theory on the world sheet of the non-Abelian semilocal string. Proceeding to $(N + \tilde{N} - 1)$ complex independent variables b_i, and φ_j,

$$b_i = \frac{n_i}{n_N}, \qquad \varphi_j = z_j \, n_N, \qquad i = 1, ..., (N-1), \qquad j = 1, ..., \tilde{N} \tag{3.11}$$

(c.f. (2.50)) we can write down the Kähler potential for the theory (3.9) in the form

$$\begin{aligned} K_{\rm eff}(b_i, \varphi_j, \bar{b}_i, \bar{\varphi}_j) &= \sum_{j=1}^{\tilde{N}} \left(|\varphi_j|^2 + \sum_{i=1}^{N-1} |(\varphi_j b_i)|^2 \right) + \frac{4\pi}{g^2} \ln\left(1 + \sum_{i=1}^{N-1} |b_i|^2 \right) \\ &\equiv |\zeta|^2 + \frac{4\pi}{g^2} \ln\left(1 + \sum_{i=1}^{N-1} |b_i|^2 \right), \end{aligned} \tag{3.12}$$

where we defined

$$|\zeta|^2 = \sum_{j=1}^{\tilde{N}} \left(|\varphi_j|^2 + \sum_{i=1}^{N-1} |(\varphi_j b_i)|^2 \right) . \tag{3.13}$$

This Kähler potential gives us the world-sheet theory written in the geometric formulation in terms of $(N + \tilde{N} - 1)$ unconstrained complex variables. The disadvantage of this geometric formulation is that the global SU(N) symmetry is not manifest much in the same way as for CP($N-1$) model. For $\tilde{N} = 1$ the Kähler potential (3.12) describes the blow-up of $\mathbb{C}^N$ at the origin.

In Sec. 4 we will derive the world sheet theory for the case of unequal quark masses, rewrite it in terms of a U(1) gauge theory and discuss its perturbative spectrum.

4 Inclusion of Quark Masses

4.1 World-Sheet Theory

Now we assume that the quark mass differences $(m_A - m_B)$ are nonvanishing in the bulk theory (2.1). This generates a mass-dependent potential on the non-Abelian semilocal string world sheet [29]. In addition, a natural IR cutoff appears which converts (1.2) in (1.3).

The leading term in this potential contains the IR logarithm,

$$V_{\rm eff} = V_{\rm eff}^{\rm IR-log} + V_{\rm eff}^{\rm finite} . \tag{4.1}$$

The first term here was calculated in [29] in the case $N = 2$. We briefly review this calculation and then generalize it to arbitrary N.

Consider first $\tilde{N} = 1$. The IR-logarithmic contribution to the potential arises from the last term in the bulk action (2.1) with $A = N + 1$,

$$\int d^4x \left| (\Phi + m_{N+1})\, q^{N+1} \right|^2 , \tag{4.2}$$

where Φ can be replaced by its vacuum expectation value (VEV) (2.7) with the logarithmic accuracy. Substituting the string solution for the extra flavor q^{N+1} (2.23) and using (2.26) and (2.27) we get

$$\int d^4x \sum_{i=1}^{N} |m_i - m_{N+1}|^2 \, |n_i|^2 \, \xi \, \frac{|\rho|^2}{r^2} . \tag{4.3}$$

The integral over r in the perpendicular plane gives the IR logarithm,

$$V_{\rm eff}^{\rm IR-log} = 2\pi\xi \int d^2x \ln\left(\frac{1}{|\Delta m||\rho|}\right) \sum_{i=1}^{N} |\rho|^2 \, |m_i - m_{N+1}|^2 \, |n_i|^2. \qquad (4.4)$$

Here we replaced the IR cutoff L with $1/\Delta m$, which is a typical scale of quark mass differences, $\Delta m \sim (m_A - m_B)$. The reason for this is that at $(m_A - m_B) \neq 0$ the Higgs branch of the theory is lifted and we do not have massless squarks in the bulk theory. All profile functions for the string solution are modified at large $r \geq |\Delta m|^{-1}$ acquiring an exponential fall-off $\sim \exp\left(-\,|\Delta m|\, r\right)$ [29]. Using the variable z (3.2) we can rewrite (4.4) as

$$V_{\rm eff} = \int d^2x \sum_{i=1}^{N} |z|^2 \, |m_i - m_{N+1}|^2 \, |n_i|^2 + V_{\rm eff}^{\rm finite} \, . \qquad (4.5)$$

Now we will follow the same logic that lead us to the exact world-sheet kinetic terms (3.5). Namely, to determine the finite part of the potential in (4.5) we take the limit $\rho \to 0$. In this limit the semilocal string reduces to the local non-Abelian string. Its potential on the world sheet is given by the twisted mass terms of the CP$(N-1)$ model [13, 14], see also the review [3]. The result for the logarithmic part can be rewritten in terms of z, as was done in Sec. 3. In this way we arrive at

$$\begin{aligned} V_{\rm eff} \;&=\; \int d^2x \left\{ \sum_{i=1}^{N} |z|^2 \, |m_i - m_{N+1}|^2 \, |n_i|^2 \right. \\ &+\; \left. \frac{4\pi}{g^2} \left[\sum_{i=1}^{N} |m_i - m|^2 \, |n_i|^2 - \left| \sum_{i=1}^{N} (m_i - m)\, |n_i|^2 \right|^2 \right] \right\}, \end{aligned} \qquad (4.6)$$

where m is the average of the first N quark masses,

$$m \equiv \frac{1}{N} \sum_{i=1}^{N} m_i \, . \qquad (4.7)$$

Generalization of (4.6) to the case $\tilde{N} > 1$ is straightforward. Our final result for the bosonic action of the world-sheet theory for the non-Abelian

semilocal string is

$$
\begin{aligned}
S_{\text{eff}} &= \int d^2x \left\{ |\partial_k (z_j n_i)|^2 + \frac{4\pi}{g^2}\left[|\partial_k n_i|^2 + (n_i^* \partial_k n_i)^2 \right] + |m_i - m_j|^2 \, |z_j|^2 |n_i|^2 \right. \\
&+ \left. \frac{4\pi}{g^2} \left[\sum_{i=1}^{N} |m_i - m|^2 \, |n_i|^2 - \left| \sum_{i=1}^{N} (m_i - m) \, |n_i|^2 \right|^2 \right] \right\}, \qquad (4.8)
\end{aligned}
$$

where m_j ($j = 1, ..., \tilde{N}$) denote masses of the last $\tilde{N}$ quarks of the bulk theory.

This theory is exact in the limit of the large IR logarithm in the same sense as in Sec. 3. The only approximation we use is the condition (3.8) which is obviously satisfied once $g\sqrt{\xi}$ is considered as an ultraviolet cutoff for the theory (4.8). In particular, we assume that

$$
|m_A| \ll g\sqrt{\xi}, \qquad A = 1, ..., N_f \,. \qquad (4.9)
$$

The model (4.8) has a hidden U(1) gauge (local) symmetry,

$$
n_i \to e^{i\alpha} \, n_i, \qquad z_j \to e^{-i\alpha} \, z_j \qquad (4.10)
$$

and therefore the number of (real) degrees of freedom is

$$
2(N + \tilde{N}) - 1 - 1 = 2(N + \tilde{N} - 1)\,, \qquad (4.11)
$$

where we subtracted two degrees of freedom associated with the condition (2.24), as well as one U(1) phase (4.10), from the total number of components of n_i and z_j.

4.2 Gauge formulation

$\mathcal{N} = (2,2)$ supersymmetric CP($N-1$) model can be nicely formulated in terms of a U(1) gauge theory in the limit of the strong gauge coupling. In this limit gauge fields and their superpartners become auxiliary [42,43]. Following the same line of reasoning we consider the local symmetry (4.10) as a gauge symmetry and rewrite the theory (4.8) as

$$
\begin{aligned}
S_{\text{eff}} &= \int d^2x \left\{ |\partial_k (z_j n_i)|^2 + |\nabla_k n_i|^2 + \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2} \, |\partial_k \sigma|^2 \right. \\
&+ \left. |m_i - m_j|^2 \, |z_j|^2 |n_i|^2 + \left| \sqrt{2}\sigma + m_i \right|^2 |n_i|^2 + \frac{e^2}{2} \left(|n_i|^2 - \frac{4\pi}{g^2} \right)^2 \right\}, \\
& i = 1, ..., N\,, \qquad j = 1, ..., \tilde{N}\,, \qquad (4.12)
\end{aligned}
$$

where the covariant derivatives are defined as

$$\nabla_k = \partial_k - iA_k\,. \tag{4.13}$$

It is assumed that at the very end we take limit $e^2 \to \infty$.

Note, that we rescale fields n_i and z_j in (4.12), which leads to the following D-term condition:

$$|n_i|^2 = \frac{4\pi}{g^2} \tag{4.14}$$

(in the limit $e^2 \to \infty$), instead of (2.24). Moreover, in this limit the gauge field A_k and its $\mathcal{N} = 2$ bosonic superpartner σ become auxiliary and can be eliminated,

$$A_k = -i\, n_i^* \partial_k n_i, \qquad \sqrt{2}\sigma = -\sum_i m_i\, |n_i|^2. \tag{4.15}$$

The global symmetry of the world sheet theory (4.12) is (the same as in the bulk theory, see (2.12))

$$\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N}) \times \mathrm{U}(1) \tag{4.16}$$

broken down to $\mathrm{U}(1)^{(N_f-1)}$ by the (s)quark mass differences.

4.3 An Alternative Gauge Formulation

The gauge formulation described in Sec. 4.2 is simple, but it has the disadvantage of including a nonstandard kinetic term which is quartic in fields. In this section we propose an alternative formulation in terms of a gauged linear sigma model with the standard kinetic terms which reduce, at low energies, to the models (4.8) and (4.12). The model presented this section can be considered as an UV completion of the model (4.12). This can be achieved at a price of including a potential term.

4.3.1 $N = 2,\ \tilde{N} = 1$

For the sake of clarity, let us start from the simplest case. We will extend the construction to the most general case in Sec. 4.3.2. The U(1) gauged linear

sigma model has the following action:

$$\begin{aligned} S_{\text{eff}} &= \int d^2x \left\{ |\partial_k Z_i|^2 + |\nabla_k n_i|^2 + \left|\bar{\nabla}_k R\right|^2 + \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2}\ |\partial_k \sigma|^2 \right. \\ &+ \left|\sqrt{2}\sigma + m_i\right|^2 |n_i|^2 + \left|\sqrt{2}\sigma\right|^2 |R|^2 + \\ &+ \left. \frac{e^2}{2}\left(|n_i|^2 - |R|^2 - \frac{4\pi}{g^2}\right)^2 + V_F(Z_A, n_i, R) \right\}, \\ & i = 1, 2. \end{aligned} \tag{4.17}$$

where the Z_i are neutral scalars, the fields n_i have charge $+1$ and the field R have charge -1. The gauge covariant derivative acting on R is, thus,

$$\bar{\nabla}_k = \partial_k + iA_k \, . \tag{4.18}$$

The first term in the third line is the D-term required by supersymmetry while the second one

$$\begin{aligned} V_F(Z_A, n_i, R_j) &= |M|^2 (|Z_1 n_2 - Z_2 n_1|^2 + |n_i|^2 |R|^2 + |Z_i|^2 |R|^2) \\ &+ |m_i - m|^2 |Z_i|^2 \end{aligned} \tag{4.19}$$

is a judiciously chosen F-term potential which comes from the superpotential

$$W_F(Z_i, n_i, R_j) = M\,(Z_1 n_2 - Z_2 n_1)\,R + \frac{1}{2}\,(m_i - m)\,Z_i^2\,, \tag{4.20}$$

where M is an auxiliary mass parameter (a UV parameter), to be sent to infinity. Note that the coefficients M and $m_i - m$ act now as complex masses for all fields, while previously we introduced the twisted masses m_i only for the n_i fields.

We now take the limit

$$e^2,\ M \to \infty\,, \tag{4.21}$$

and integrate out massive fields (with masses of order e, M). Integrating out the scalar fields we obtain the following vacuum equations:

$$R = 0, \quad Z_1 n_2 = Z_2 n_1 \, . \tag{4.22}$$

Moreover, in the limit above, the D-term condition

$$|n_i|^2 - |R|^2 = |n_i|^2 = \frac{4\pi}{g^2} \tag{4.23}$$

is implemented.[6] The gauge field A_k and its $\mathcal{N}=(2,2)$ bosonic superpartner σ become auxiliary and can be eliminated too,

$$\begin{aligned} A_k &= -i\, n_i^* \partial_k n_i + i\, R^* \partial_k R = -i\, n_i^* \partial_k n_i \,, \\ \sqrt{2}\sigma &= -\frac{\sum_i m_i |n_i|^2}{\sum_i |n_i|^2 + |R|^2} = -\sum_i m_i |n_i|^2 \,. \end{aligned} \tag{4.24}$$

Substituting the relations above into (4.17) we obtain:

$$\begin{aligned} S_{\text{eff}} &= \int d^2x \left\{ |\partial_k Z_1|^2 + \left| \partial_k \left(Z_1 \frac{n_2}{n_1} \right) \right|^2 + \frac{4\pi}{g^2} \left[|\partial_k n_i|^2 + (n_i^* \partial_k n_i)^2 \right] \right. \\ &+ \frac{4\pi}{g^2} \left[\sum_{i=1}^{N} |m_i - m|^2\, |n_i|^2 - \left| \sum_{i=1}^{N} (m_i - m)\, |n_i|^2 \right|^2 \right] \\ &+ \left. \sum_{i=1}^{N} |m_i - m|^2 |Z_1|^2 + \sum_{i=1}^{N} |m_i - m|^2 \left| Z_1 \frac{n_2}{n_1} \right|^2 \right\}, \end{aligned} \tag{4.25}$$

which exactly reduces to the theory written in (4.8) with the identification

$$Z_1 \equiv z\, n_1 \tag{4.26}$$

---

[6]It is important that $R = 0$ in the vacuum, see (4.22).

4.3.2 N arbitrary, $\tilde{N} > 1$

Essentially the same construction as in Sec. 4.3.1 can be can be carried out in the most general case. Consider the following gauged sigma model:

$$\begin{aligned} S_{\rm eff} &= \int d^2x \left\{ |\partial_k Z_A|^2 + |\nabla_k n_i|^2 + \left|\bar{\nabla}_k R_B\right|^2 + \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2}\, |\partial_k \sigma|^2 \right. \\ &+ \left|\sqrt{2}\sigma + m_i\right|^2 |n_i|^2 + \left|\sqrt{2}\sigma\right|^2 |R_B|^2 + \\ &+ \left. \frac{e^2}{2}\left(|n_i|^2 - |R_B|^2 - \frac{4\pi}{g^2}\right)^2 + V_F(Z_A, n_i, R_B) \right\}, \\ & A = 1, ..., \tilde{N}N\,, \quad i = 1, ..., N\,, \quad B = 1, ..., \tilde{N}(N-1)\,. \end{aligned} \tag{4.27}$$

Note that we introduced a large set of new charge-zero fields Z_A and negatively charged fields R_B. The theory in Eq. (4.27) includes the potential $V_F(Z_A, n_i, R_B)$,

$$\begin{aligned} V_F(Z_A, n_i, R_B) &= |M|^2 \sum_{o=1}^{N-1} \sum_{p=1}^{\tilde{N}} |Z_{N(p-1)+1} n_{o+1} - Z_{N(p-1)+o+1} n_1|^2 + \\ &+ |M|^2 |\sum_{o=1}^{N-1} \sum_{p=1}^{\tilde{N}} Z_{N(p-1)+o+1}\, R_{(N-1)(p-1)+o}|^2 + \\ &+ |M|^2 \sum_{o=1}^{N-1} |\sum_{p=1}^{\tilde{N}} Z_{N(p-1)+1}\, R_{(N-1)(p-1)+o}|^2 + \\ &+ |M|^2 \sum_{p=1}^{\tilde{N}} |\sum_{o=1}^{N-1} n_{o+1}\, R_{(N-1)(p-1)+o}|^2 + \\ &+ |M|^2 \sum_{o=1}^{N-1} \sum_{p=1}^{\tilde{N}} |n_1 R_{(N-1)(p-1)+o}|^2 + \\ &+ \sum_{j=1}^{\tilde{N}} \sum_{i=1}^{N} |m_i - m_j|^2 |Z_{N(j-1)+i}|^2 + \\ &+ |M|^2 \sum_B |R_B|^2\,. \end{aligned} \tag{4.28}$$

This potential is consistent with $\mathcal{N} = (2,2)$ supersymmetry since it comes from the following superpotential:

$$\begin{aligned} W_F(Z_A, n_i, R_B) &= M \sum_{o=1}^{N-1} \sum_{p=1}^{\tilde{N}} R_{(N-1)(p-1)+o} \big(Z_{N(p-1)+1} n_{o+1} \\ &- Z_{N(p-1)+o+1} n_1 \big) + \\ &+ \frac{1}{2} \sum_{j=1}^{\tilde{N}} \sum_{i=1}^{N} (m_i - m_j) Z^2_{N(j-1)+i} + \frac{1}{2} M \sum_B R_B^2 \, . \end{aligned} \tag{4.29}$$

After some straightforward but rather tedious algebra one can show that vanishing of the potential (4.28) requires vanishing of all R fields,

$$R_B = 0 \, , \quad \forall \, B \, , \tag{4.30}$$

and that the nontrivial constraints on the fields are given by imposing the vanishing of the first line in (4.28),

$$Z_{N(p-1)+o+1} = Z_{N(p-1)+1} \frac{n_{o+1}}{n_1}, \quad o = 1, \ldots, N-1, \quad p = 1, \ldots, \tilde{N} \, . \tag{4.31}$$

Using the relations above and the identifications

$$Z_{N(j-1)+1} \equiv z_j \, n_1 \tag{4.32}$$

we arrive at

$$\begin{aligned} \sum_A |\partial_k Z_A|^2 &= \sum_{i,j} |\partial_k (z_j n_i)|^2, \\ \sum_{i,j} |m_i - m_j|^2 |Z_{N(j-1)+i}|^2 &= \sum_{i,j} |m_i - m_j|^2 |z_j|^2 |n_i|^2 \, . \end{aligned} \tag{4.33}$$

This, in conjunction with the condition $R_B = 0$ in Eq. (4.27), leads us to (4.8) again.

5 Quasiclassical spectrum

In this section we will analyze the vacuum structure and the mass spectrum in our world-sheet zn theory. It is simpler to obtain it from the action (4.12) written in the gauged formulation. In this paper we will limit ourselves to the quasiclassical study of the theory (4.12) leaving its investigation at the quantum level for future work. First we will consider perturbative spectrum.

5.1 Perturbative Spectrum

If all quark masses are different, the theory (4.12) has N isolated vacua at

$$\sqrt{2}\sigma = -m_{i_0}, \qquad n_{i_0} = \sqrt{\frac{4\pi}{g^2}}, \qquad n_{i\neq i_0} = 0, \qquad z_j = 0, \tag{5.1}$$

where i_0 can acquire any value,

$$i_0 = 1, ..., N\,, \;\; \text{while} \;\; j = 1, ..., \tilde{N}\,. \tag{5.2}$$

The above vacua of the world-sheet theory correspond to N elementary non-Abelian strings. The spectrum of $n_{i\neq i_0}$ and z_j excitations can be read-off from the action (4.12),

$$m_{n_i} = m_i - m_{i_0}, \qquad i \neq i_0, \qquad m_{z_j} = m_j - m_{i_0}. \tag{5.3}$$

Now suppose that one of the masses of the first N quarks coincides with another mass of the last $\tilde{N}$ quarks, $m_{j_0} = m_{i_0}$. Then the theory develops a noncompact Higgs branch growing from the vacuum at $\sqrt{2}\sigma = -m_{i_0}$, namely,

$$\sqrt{2}\sigma = -m_{i_0}, \quad n_{i_0} = \sqrt{\frac{4\pi}{g^2}}, \quad n_{i\neq i_0} = 0, \quad z_{j\neq j_0} = 0\,, \qquad z_{j_0} = z_0\,, \tag{5.4}$$

where z_0 is an arbitrary complex number. The (real) dimension of this Higgs branch is $\dim \mathcal{H} = 2$.

Although both kinetic and mass terms in (4.12) acquire a dependence on z_0 the masses of $n_{i\neq i_0}$ and z_j excitations remain the same, they are given by (5.3). It is only the field z_{j_0} that becomes massless; it corresponds to fluctuations along the Higgs branch.

Now, let us go to very low energies, much lower than the quark mass differences $|\Delta m|$. Then, the low-energy effective theory on the Higgs branch is just a trivial free-field theory for the massless complex field z_{j_0},

$$S_{\text{Higgs branch}} = \int d^2x \, |\partial_k z_{j_0}|^2 \,. \tag{5.5}$$

If more than one masses of the first N quarks coincide with certain masses of the last $\tilde{N}$ quarks, more noncompact Higgs branches develop. These Higgs branches are not lifted in quantum theory. In contrast, the compact Higgs branches which classically develop provided that several masses of first quarks coincide with each other are lifted in quantum theory much in the same way as in CP$(N-1)$ model.

5.2 Semiclassical kink spectrum

In addition to perturbative excitations, the theory (4.12) supports BPS kinks interpolating between different vacua. Let us calculate their masses in the quasiclassical approximation. To do so we write down the Bogomol'nyi representation for the kink energy. Assuming for simplicity that the quark masses and σ are real and that all fields depend only on x_3 we can rewrite (4.12) in the limit $e^2 \to \infty$ as follows:

$$\begin{aligned} E_{\text{kink}} &= \int dx_3 \left\{ |\partial_{x_3}(z_j n_i)|^2 + |\nabla_{x_3} n_i|^2 + |m_i - m_j|^2 \, |z_j|^2 |n_i|^2 \right. \\ &+ \left. \left| \sqrt{2}\sigma + m_i \right|^2 |n_i|^2 \right\} \\ &= \int dx_3 \left\{ |\partial_{x_3}(z_j n_i) + (m_i - m_j)\, z_j \, n_i|^2 + \left| \nabla_{x_3} n_i + (\sqrt{2}\sigma + m_i)\, n_i \right|^2 \right. \\ &+ \left. \frac{4\pi}{g^2} \sqrt{2} \partial_{x_3} \sigma \right\}, \end{aligned} \tag{5.6}$$

where we use the constraint (4.14) and dropped the boundary terms

$$(\sqrt{2}\sigma + m_i)\, |n_i|^2 \;\; \text{and} \;\; (m_i - m_j)\, |z_j|^2 |n_i|^2 \,. \tag{5.7}$$

In particular, the last one vanishes at generic masses in all vacua (5.1), because $z_j = 0$, while on the Higgs branches (5.4) it is zero because $m_{i_0} = m_{j_0}$.

From the Bogomol'nyi representation (5.6) we see that the kink profile functions satisfy the first-order equations

$$\partial_{x_3}(z_j n_i) + (m_i - m_j)\, z_j\, n_i = 0,$$

$$\nabla_{x_3} n_i + (\sqrt{2}\sigma + m_i)\, n_i = 0, \tag{5.8}$$

while the kink masses are given by the boundary term in (5.6). In particular, the mass of the kink interpolating between the "neighboring" vacua i_0 and $i_0 + 1$ is

$$\begin{aligned} m^{\rm kink}_{i_0 \to i_0+1} &= \left| \frac{4\pi}{g^2}\, \sqrt{2} \Big[\sigma(x_3 = \infty) - \sigma(x_3 = -\infty) \Big] \right| \\ &= \left| \frac{4\pi}{g^2}\, \left(m_{i_0} - m_{i_0+1} \right) \right| . \end{aligned} \tag{5.9}$$

For generic masses the solution of the first-order equations (5.8) is particularly simple. The first equation is solved by $z_j = 0$, while the second one reduces to the first-order equation for BPS kinks in the CP($N-1$) model with twisted masses [19]. Thus, the kinks' profile functions are the same as in the CP($N-1$) model.

We recall that the monopoles are confined in the bulk theory in the Higgs vacuum (2.7). In fact, in the U(N) gauge theories they are presented by junctions of two different elementary non-Abelian strings. Since N elementary non-Abelian strings correspond to N vacua of the world-sheet theory, the confined monopoles of the bulk theory are seen as kinks in the world-sheet theory [18, 13, 14].

As was shown in [19], the BPS spectrum of dyons (at the singular point on the Coulomb branch in which N quarks become massless) in the four-dimensional bulk theory (2.1), for $N_f = N$, identically coincides with the BPS spectrum in the two-dimensional twisted-mass deformed CP($N-1$) model. The reason for this coincidence was revealed in [13, 14]. Although the 't Hooft–Polyakov monopole on the Coulomb branch looks very different from the string junction of the theory in the Higgs regime, amazingly, their masses are the same [13,14]. This is due to the fact that the mass of the BPS states (the string junction is a 1/4-BPS state) cannot depend on ξ because ξ is a nonholomorphic parameter. Since the confined monopole emerges in the world-sheet theory as a kink, the Seiberg–Witten formula for its mass

should coincide with the exact result for the kink mass in two-dimensional $\mathcal{N}=2$ twisted-mass deformed CP$(N-1)$ model, which is the world-sheet theory for the non-Abelian string in the bulk theory with $N_f = N$. Thus, we arrive at the statement of coincidence of the BPS spectra in both theories.

Clearly the same correspondence should be true also in the $N_f > N$ case. Let us verify the coincidence of the BPS spectra of the bulk and world-sheet theories in the quasiclassical approximation. Taking the limit $\xi \to 0$ in (2.5) we see that, in the vacuum (2.7), the massive gauge bosons and first N quarks have masses

$$m_{N\times N} = m_i - m_{i'}, \qquad i, i' = 1, ..., N, \qquad i \neq i', \tag{5.10}$$

while the last $\tilde{N}$ quarks

$$m_{N\times\tilde{N}} = m_i - m_j, \qquad i = 1, ..., N, \qquad j = 1, ..., \tilde{N}. \tag{5.11}$$

We see that this spectrum is identical to the perturbative spectrum of the world-sheet theory (5.3).

The monopole spectrum of the bulk theory is given by the Seiberg–Witten formula [7]

$$m_{\text{monopole}} = |\vec{a}_D\, \vec{n}_m| \approx \left|\frac{4\pi}{g^2}\, \vec{a}\, \vec{n}_m\right|, \tag{5.12}$$

where we use the quasiclassical approximation. Moreover, $\vec{a}$ represents diagonal entries of the adjoint field Φ while $\vec{a}_D$ stands for corresponding dual potentials and $\vec{n}_m$ is the magnetic charge of a monopole. In particular, for the elementary monopoles $\vec{n}_m = (0, ..., 1, -1, 0, ...)$ (with nonvanishing entries at the i-th and $(i+1)$-th positions) we get

$$m_{\text{monopole}} \approx \left|\frac{4\pi}{g^2}\, (m_i - m_{i+1})\right|, \qquad i = 1, ..., N-1\,, \tag{5.13}$$

where we use (2.7). These masses coincides with the kink masses (5.9) of the world-sheet theory in the quasiclassical approximation. Explicit verification that the exact BPS spectra of both theories agree is left for future work.

6 Vortices from D-Branes: Comparing with Hanany and Tong

6.1 Weighted CP($N_f - 1$) model

As was mentioned in Sec. 1, non-Abelian semilocal strings were analyzed previously [11, 14] within a complementary approach based on D-branes. To make contact with field theory it is highly instructive to compare our field-theoretic results with those obtained by Hanany and Tong. They conjectured that the effective theory on the world sheet of the non-Abelian semilocal string is given by the weighted CP($N_f - 1$) model. The latter can be represented as a strong-coupling limit ($e^2 \to \infty$) of the two-dimensional U(1) gauge theory with N positive and $\tilde{N}$ negative charges, namely

$$
\begin{aligned}
S_{\rm HT} &= \int d^2x \left\{ |\nabla_k n_i^w|^2 + |\tilde{\nabla}_k z_j^w|^2 + \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2} \, |\partial_k \sigma|^2 \right. \\
&+ \left| \sqrt{2}\sigma + m_i \right|^2 |n_i^w|^2 + \left| \sqrt{2}\sigma + m_j \right|^2 \left| z_j^w \right|^2 \\
&+ \left. \frac{e^2}{2} \left(|n_i^w|^2 - |z_j^w|^2 - \frac{4\pi}{g^2} \right)^2 \right\}, \\
& \quad i = 1, ..., N, \qquad j = 1, ..., \tilde{N}\,, \qquad \tilde{\nabla}_k = \partial_k + iA_k\,. \qquad (6.1)
\end{aligned}
$$

With respect to the U(1) gauge field, the fields n_i^w and z_i^w have charges $+1$ and -1, respectively. We endow these fields with a superscript "w" (weighted) to distinguish them from the n_i and z_j fields which appear in our world-sheet zn theory (4.12). If only the charge $+1$ fields were present, in the limit $e^2 \to \infty$ we would get a conventional twisted-mass deformed CP($N - 1$) model.

6.2 Quasiclassical spectrum

Although the weighted CP($N_f - 1$) model and the zn model look quite different we will show momentarily that the quasiclassical spectra of excitations of these two models are the same. Let us start from the perturbative spectrum.

From (6.1) we see that the Hanany–Tong world-sheet theory has N vacua

(i.e. N strings from the standpoint of the bulk theory),

$$\sqrt{2}\sigma = -m_{i_0}, \qquad n^w_{i_0} = \sqrt{\frac{4\pi}{g^2}}\,, \qquad n^w_{i\neq i_0} = z^w_j = 0\,, \tag{6.2}$$

where $i_0 = 1, ..., N$.

In each vacuum there are $2(N_f - 1)$ elementary excitations, counting real degrees of freedom, much in the same way as in (4.12). The action (6.1) contains N complex fields n^w_i and $\tilde{N}$ complex fields z^w_j. The phase of $n^w_{i_0}$ is eaten by the Higgs mechanism. The condition $|n^w_{i_0}|^2 = \frac{4\pi}{g^2}$ eliminates one extra field. The physical masses of the elementary excitations

$$m_{n^w_i} = m_i - m_{i_0}, \qquad i \neq i_0, \qquad m_{z^w_j} = m_j - m_{i_0}. \tag{6.3}$$

This spectrum is identical to the perturbative spectrum of the zn model (4.12)

Now, suppose again that $m_{j_0} = m_{i_0}$. Then the theory (6.1) also develops a noncompact Higgs branch growing from the vacuum at $\sqrt{2}\sigma = -m_{i_0}$, namely

$$\sqrt{2}\sigma = -m_{i_0}, \qquad |n^w_{i_0}|^2 - |z^w_{j_0}|^2 = \frac{4\pi}{g^2}, \qquad n^w_{i\neq i_0} = 0, \qquad z^w_{j\neq j_0} = 0. \tag{6.4}$$

The (real) dimension of this Higgs branch is two, much in the same way as for the Higgs branch in the world-sheet theory (4.12).

Moreover, the spectrum of fields $n^w_{i\neq i_0}$ and $z^w_{j\neq j_0}$ is still given by (6.3). One degree of freedom of two complex fields $n^w_{i_0}$ and $z^w_{j_0}$ is eaten by the Higgs mechanism, while the other is fixed by the second constraint in (6.4). The remaining two degrees of freedom are massless. They correspond to fluctuations along the Higgs branch.

We see that the perturbative spectra of these two models (4.12) and (6.1) are identical.

Now consider the effective low-energy theory on the Higgs branch (6.4). At energies below the quark mass differences only the fields $n^w_{i_0}$ and $z^w_{j_0}$ are relevant. We resolve the constraint in the second equation in (6.4) by writing

$$n^w_{i_0} = \sqrt{\frac{4\pi}{g^2}}\, e^{i\alpha + i\beta}\, \cosh w, \qquad z^w_{j_0} = \sqrt{\frac{4\pi}{g^2}}\, e^{i\alpha - i\beta}\, \sinh w, \tag{6.5}$$

where we introduced two phases for two complex fields. From the action (6.1) we find the gauge potential

$$A_k = 2\left(\partial_k\alpha + \frac{\partial_k\beta}{\cosh 2w}\right). \tag{6.6}$$

Substituting this together with (6.5) into the action (6.1) we get [44]

$$S^{\rm HT}_{\rm Higgs\ branch} = \int d^2x \cosh 2w \left\{ (\partial_k w)^2 + (\partial_k \beta)^2 \tanh^2 2w \right\}, \tag{6.7}$$

where the common phase α is eaten by the Higgs mechanism, and we are left with a sigma model with two real degrees of freedom.

This theory on the Higgs branch is clearly different from the free theory (5.5). The target space in (6.7) is hyperboloid with a nonvanishing curvature. This shows that two models, (4.12) on the one hand and (6.1) on the other are different, despite the coincidence of their spectra.[7]

Now let us briefly review the kink spectrum of the weighted CP$(N_f - 1)$ model in the quasiclassical approximation. Assuming again the quark masses and σ to be real and all fields depend only on x_3 we cast the Bogomol'nyi representation for the kink energy in the model (6.1) in the limit $e^2 \to \infty$ in the form

$$\begin{aligned} E_{\rm kink} &= \int dx_3 \left\{ |\nabla_{x_3} n_i^w|^2 + \left|\nabla_{x_3} z_j^w\right|^2 + \left|\sqrt{2}\sigma + m_i\right|^2 |n_i^w|^2 \right. \\ &+ \left. \left|\sqrt{2}\sigma + m_j\right|^2 \left|z_j^w\right|^2 \right\} \\ &= \int dx_3 \left\{ \left|\nabla_{x_3} n_i^w + (\sqrt{2}\sigma + m_i)\, n_i^w \right|^2 + \left|\bar\nabla_{x_3} z_j^w - (\sqrt{2}\sigma + m_j)\, z_j^w\right|^2 \right. \\ &+ \left. \frac{4\pi}{g^2}\, \sqrt{2}\partial_{x_3}\sigma \right\}. \end{aligned} \tag{6.8}$$

This representation shows that kink solutions satisfy the first-order equations

$$\begin{aligned} &\nabla_{x_3} n_i^w + (\sqrt{2}\sigma + m_i)\, n_i^w = 0, \\ &\bar\nabla_{x_3} z_j^w - (\sqrt{2}\sigma + m_j)\, z_j^w = 0, \end{aligned} \tag{6.9}$$

while the kink masses are given by the boundary term in (6.8). Much in the same way as in the theory (4.12) this gives, for the kink interpolating

[7] Interrelation between aspects of the Hanany–Tong model and field-theoretic predictions for non-Abelian strings was previously studied in [45] in the case of two coaxial strings. There, it was found that a limited number of "protected" quantities, such as the BPS spectra, agree, while others (e.g. the metric) disagree.

between vacua i_0 and i_0+1,

$$m^{\text{kink}}_{i_0\to i_0+1} = \left|\frac{4\pi}{g^2}\,\left(m_{i_0} - m_{i_0+1}\right)\right|. \tag{6.10}$$

Again, the kink spectrum we get is identical to that in (5.9) obtained in the world-sheet theory (4.12).

In Sec. 6.3 we will show that geometries of the target spaces of these two models are different (in the case when all fields are relevant). Given the agreement of the BPS spectra this might seem surprising. Maybe not (cf. [45]). Such a situation could have a simple explanation. While the Kähler potentials of two $\mathcal{N}=(2,2)$ supersymmetric sigma models are different their effective twisted superpotentials could agree. This would ensure the coincidence of their BPS spectra.

The exact BPS spectrum in the weighted CP(N_f-1) model (6.1) was originally discussed in [43]. It was shown to agree with the BPS spectrum of the bulk theory in the vacuum (2.7) on the *Coulomb branch* (i.e. at $\xi\to 0$) [20]. This was considered to be a strong argument supporting the conjecture that the weighted CP(N_f-1) model (6.1) fully presents a correct world-sheet theory on the semilocal non-Abelian string [14, 29]. Now we are certain that this conjecture is not correct. Although the BPS spectrum of weighted CP(N_f-1) model (6.1) coincides with that in the bulk theory, this model is different from the genuine world-sheet theory on the semilocal non-Abelian string, the *zn* model (4.12).

6.3 Comparing two metrics

The Kähler potential of the theory (6.1) can be written, using the superfield formalism, in the following simple form:

$$\begin{aligned} K_{\text{HT}} &= e^{-V}|n_i^w|^2 + e^{V}|z_j^w|^2 + \frac{4\pi}{g^2}V, \\ i &= 1,\dots,N, \qquad j=1,\dots,\tilde{N}\,, \end{aligned} \tag{6.11}$$

where n_i^w and z_j^w are chiral superfields and V is a vector superfield and summations over indices i and j are implicit. We can eliminate V by solving

its equations of motion

$$\begin{aligned}\partial_V K_{\rm HT} &= -e^{-V}|n_i^w|^2 + e^V|z_j^w|^2 + \frac{4\pi}{g^2} = 0\,; \\ & |n_i^w|^2 e^{-2V} - \frac{4\pi}{g^2} e^{-V} - |z_j^w|^2 = 0\,. \end{aligned} \tag{6.12}$$

By virtue of the D-term condition, we can assume $|n_i^w|^2 \neq 0$ whenever $4\pi/g^2 > 0$, then

$$e^{-V} = \frac{1}{2|n_i^w|^2}\left(\frac{4\pi}{g^2} + \sqrt{\left(\frac{4\pi}{g^2}\right)^2 + 4|n_i^w|^2|z_j^w|^2}\,\right)\,. \tag{6.13}$$

Substituting this expression back in the Kähler potential, we obtain, up to Kähler transformations, the exact expression

$$\begin{aligned} K_{\rm HT} &= \frac{1}{2}\left(\frac{4\pi}{g^2} + \sqrt{\left(\frac{4\pi}{g^2}\right)^2 + 4|M_{ij}|^2}\right) + \frac{2|M_{ij}|^2}{\frac{4\pi}{g^2} + \sqrt{\left(\frac{4\pi}{g^2}\right)^2 + 4|M_{ij}|^2}} \\ &- \frac{4\pi}{g^2}\ln\left(\frac{4\pi}{g^2} + \sqrt{\left(\frac{4\pi}{g^2}\right)^2 + 4|M_{ij}|^2}\right) + \frac{4\pi}{g^2}\ln\left(1 + \frac{|M_{i1}|^2}{|M_{N1}|^2}\right)\,, \end{aligned} \tag{6.14}$$

where we defined the meson fields as

$$M_{ij} = n_i^w z_j^w\,. \tag{6.15}$$

Note that not all of the $N \times \tilde{N}$ mesonic fields are independent because of the relations

$$M_{ij}M_{kl} = M_{kj}M_{il}\,.$$

The total number of independent complex degree of freedoms is $N + \tilde{N} - 1$, which is the total number of fields in the theory minus one complex rescaling of the fields. We can choose the following set of independent mesons:

$$M_{i1} = n_i^w z_1^w\,, \quad M_{Nj} = n_N^w z_j^w\,, \quad i \neq N\,. \tag{6.16}$$

All other mesons are given by the formula

$$M_{ij} = M_{i1} M_{Nj} / M_{N1} \,. \tag{6.17}$$

The combination $|M_{ij}|^2$ can the be written as

$$\sum_{i,j=1}^{N,\tilde{N}} |M_{ij}|^2 = \sum_{j=1}^{\tilde{N}} \left(|M_{Nj}|^2 + \sum_{i=1}^{N-1} |M_{i1}|^2 |M_{Nj}|^2 / |M_{N1}|^2 \right) . \tag{6.18}$$

To compare the expression above with the field-theoretic result (3.12), we identify the set of independent mesons used in the Kähler quotient construction with the set of moduli found in the field-theoretic derivation,

$$\varphi_j \equiv M_{Nj} = n_N^w z_j^w, \quad b_i \equiv \frac{M_{i1}}{M_{N1}} = \frac{n_i^w}{n_N^w}, \quad |\zeta|^2 \equiv |M_{ij}|^2 \,. \tag{6.19}$$

For simplicity, let us compare the two geometries, (3.12) vs. (6.14), at first order in the expansion for large g^2. The Kähler potential obtained from the Hanany–Tong model is then

$$K_{\mathrm{HT}} \;=\; 2\sqrt{|\zeta|^2} - \frac{2\pi}{g^2} \log(|\zeta|^2) + \frac{4\pi}{g^2} \log(1 + |b_i|^2) \,, \tag{6.20}$$

while the exact Kähler potential we found in field theory is

$$K_{\mathrm{eff}} \;=\; |\zeta|^2 + \frac{4\pi}{g^2} \log(1 + |b_i|^2) \,. \tag{6.21}$$

To explicitly demonstrate that the two geometries described above are indeed different, we calculate the scalars curvatures of the respective target spaces and verify that they disagree. For any Kähler manifold, the Ricci scalar can be easily calculated using the formulas

$$\begin{aligned} g_{p\bar{q}} &= \partial_p \partial_{\bar{q}} K \,, \\ R_{p\bar{q}} &= -\partial_p \partial_{\bar{q}} \left(\ln \det g_{p\bar{q}} \right) \,, \\ R &= g^{p\bar{q}} R_{p\bar{q}} \,, \qquad p, q = 1, N + \tilde{N} - 1 \,, \end{aligned} \tag{6.22}$$

where we endow the set of independent complex fields which describe the moduli space (φ_j and b_i) with indices p and q.

Start from the case $N = 2$, $\tilde{N} = 1$. Evaluating (6.22) using (6.20) and (6.21) which implies

$$|\zeta|^2 = |\varphi|^2(1 + |b|^2)\,, \tag{6.23}$$

we arrive at

$$\begin{aligned} R_{\mathrm{HT}} &= \frac{1}{\sqrt{|\zeta|^2}} - \frac{2\pi}{g^2|\zeta|^2} + \mathcal{O}(1/g^2)\,; \\ R_{\mathrm{eff}} &= 0\,. \end{aligned} \tag{6.24}$$

We thus conclude that the geometry of the target space derived from field theory has the vanishing Ricci scalar, while for geometry described by the Hanany–Tong model the Ricci scalar does not vanish, rather it falls off as $1/|\zeta|$.

In the case $N = 2 = \tilde{N} = 2$, we consider (6.20) and (6.21) with

$$|\zeta|^2 = (|\varphi_1|^2 + |\varphi_2|^2)(1 + |b|^2)\,. \tag{6.25}$$

The Ricci scalars are then

$$\begin{aligned} R_{\mathrm{HT}} &= \frac{1}{\sqrt{|\zeta|^2}} + \mathcal{O}(1/g^2)\,, \\ R_{\mathrm{eff}} &= -\frac{2}{|\zeta|^2} + \frac{8\pi}{g^2|\zeta|^4} + \mathcal{O}(1/g^2)\,. \end{aligned} \tag{6.26}$$

Disagreement is obvious. This parallels the conclusion of [45].

7 Duality

In this section we will discuss duality relation for the zn model. By no means this relation is accidental. Rather it is in one-to-one correspondence with the duality relation for the bulk theories.

7.1 Bulk Duality

As was shown in [46,47], at $\sqrt{\xi} \sim \Lambda$ the bulk theory goes through a crossover transition to the strong coupling regime. At small ξ ($\sqrt{\xi} \ll \Lambda$) this regime

can be described in terms of weakly coupled dual $\mathcal{N}=2$ SQCD, with the gauge group

$$\mathrm{U}(\tilde{N}) \times \mathrm{U}(1)^{N-\tilde{N}}\,, \tag{7.1}$$

and N_f flavors of light *dyons*. This non-Abelian $\mathcal{N}=2$ duality is, in a sense, similar to Seiberg's duality in $\mathcal{N}=1$ supersymmetric QCD [7,48], for further details see [49]. Later a dual non-Abelian gauge group $\mathrm{SU}(\tilde{N})$ was identified on the Coulomb branch at the root of a baryonic Higgs branch in the $\mathcal{N}=2$ supersymmetric $\mathrm{SU}(N)$ gauge theory with massless quarks [50].

Light dyons are in the fundamental representation of the gauge group $\mathrm{U}(\tilde{N})$ and are charged under Abelian factors in (7.1). In addition, there are light dyons D^l ($l=\tilde{N}+1,...,N$), neutral under the $\mathrm{U}(\tilde{N})$ group, but charged under the U(1) factors. A small but nonvanishing ξ triggers condensation of all these dyons,

$$\langle D^{lA}\rangle \;=\; \sqrt{\xi}\left(\begin{array}{cccccc} 0 & \ldots & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & \ldots & 1 \end{array}\right), \quad \langle \bar{\tilde{D}}^{lA}\rangle = 0, \quad l=1,...,\tilde{N},$$

$$\langle D^{l}\rangle \;=\; \sqrt{\xi}, \qquad \langle \bar{\tilde{D}}^{l}\rangle = 0\,, \qquad l=\tilde{N}+1,...,N\,. \tag{7.2}$$

Now, consider the equal quark mass case. Both, the gauge and flavor $\mathrm{SU}(N_f)$ groups, are broken in the vacuum. However, the color-flavor locked form of (7.2) guarantees that the diagonal global $\mathrm{SU}(\tilde{N})_{C+F}$ survives. More exactly, the unbroken global group of the dual theory is

$$\mathrm{SU}(N)_F \times \mathrm{SU}(\tilde{N})_{C+F} \times \mathrm{U}(1)\,, \tag{7.3}$$

the same as in the original theory, see (2.12). Here $\mathrm{SU}(\tilde{N})_{C+F}$ is a global unbroken color-flavor rotation, which involves the last $\tilde{N}$ flavors, while $\mathrm{SU}(N)_F$ factor stands for the flavor rotation of the first N dyons. Thus, a color-flavor locking takes place in the dual theory too, although in a different way. Now colors are "locked" to the last $\tilde{N}$ flavors instead of the first N, see (2.7) and (7.2).

For generic quark masses the global symmetry (2.12) is broken down to $\mathrm{U}(1)^{N_f-1}$.

7.2 Dual world-sheet theory

Much in the same way as in the original theory, the presence of the global $\mathrm{SU}(\tilde{N})_{C+F}$ group is the reason behind the formation of the non-Abelian strings. We can repeat all the steps that leads us to the effective world-sheet theory (4.12) on the non-Abelian semilocal string for the dual bulk theory. Now we have $\tilde{N}$ orientation moduli $\tilde{n}_j$ with masses $m_j = \{m_{N+1}, ..., m_{N_f}\}$ and N size moduli $\tilde{z}_i$ with masses $m_i = \{m_1, ..., m_N\}$ ($j = 1, ..., \tilde{N}$, $i = 1, ..., N$). The bosonic part of the action has the form

$$\begin{aligned} S^D_{eff} &= \int d^2x \left\{ |\partial_k(\tilde{z}_i \tilde{n}_j)|^2 + |\nabla_k \tilde{n}_j|^2 + \frac{1}{4\tilde{e}^2} F^2_{kl} + \frac{1}{\tilde{e}^2}\, |\partial_k \sigma|^2 \right. \\ &+ \left. |m_i - m_j|^2\, |\tilde{z}_i|^2 |\tilde{n}_j|^2 + \left| \sqrt{2}\sigma + m_j \right|^2 |\tilde{n}_j|^2 + \frac{\tilde{e}^2}{2} \left(|\tilde{n}_j|^2 - \frac{4\pi}{\tilde{g}^2} \right)^2 \right\}, \\ & \quad i = 1, ..., N\,, \qquad j = 1, ..., \tilde{N}\,, \end{aligned} \tag{7.4}$$

where $\tilde{g}^2$ is the dual bulk coupling, and the strong coupling limit $\tilde{e} \to \infty$ is assumed.

Classically, the vacua of this theory at generic quark masses are at

$$\sqrt{2}\sigma = -m_{j_0}, \qquad \tilde{n}_{j_0} = \sqrt{\frac{4\pi}{\tilde{g}^2}}, \qquad \tilde{n}_{j \neq j_0} = 0, \qquad \tilde{z}_i = 0, \tag{7.5}$$

where j_0 can be

$$j_0 = 1, ..., \tilde{N}\,,$$

while $i = 1, ..., N$. These vacua of the dual world-sheet theory correspond to $\tilde{N}$ elementary non-Abelian strings in the dual bulk theory.

The spectrum of $\tilde{n}_{j \neq j_0}$ and $\tilde{z}_i$ excitations is

$$m_{\tilde{n}_j} = m_j - m_{j_0}, \qquad j \neq j_0, \qquad m_{\tilde{z}_i} = m_i - m_{j_0}. \tag{7.6}$$

Note, that this spectrum is different from the perturbative spectrum of the original world-sheet theory, see (5.3).

Suppose again that one of the masses of the first N quarks coincides with another mass of the last $\tilde{N}$ quarks,

$$m_{j_0} = m_{i_0}\,.$$

Then the dual theory also develops a noncompact Higgs branch growing from the vacuum at $\sqrt{2}\sigma = -m_{j_0}$, namely,

$$\sqrt{2}\sigma = -m_{j_0}\,, \quad \tilde{n}_{j_0} = \sqrt{\frac{4\pi}{\tilde{g}^2}}\,, \quad \tilde{n}_{j\neq j_0} = 0\,, \quad \tilde{z}_{i\neq i_0} = 0\,, \quad \tilde{z}_{i_0} = \tilde{z}_0\,, \tag{7.7}$$

where $\tilde{z}_0$ is a complex number. The (real) dimension of this Higgs branch is $\dim\mathcal{H} = 2$.

Again, the masses of the $\tilde{n}_{j\neq j_0}$ and $\tilde{z}_{i\neq i_0}$ excitations remain the same, they are given in (7.6). The field $\tilde{z}_{i_0}$ becomes massless, it corresponds to fluctuations along the Higgs branch.

The quasiclassical kink spectrum for the dual world-sheet theory (7.4) can be obtained much in the same way as was done for the original world-sheet theory in Sec. 5.2. Writing down a Bogomol'nyi representation for the dual model (7.4) analogous to that in (5.6) we get the masses of the kinks interpolating between the "neighboring" vacua j_0 and j_0+1, see (7.5),

$$m^{\rm kink}_{j_0\to j_0+1} = \left|\frac{4\pi}{\tilde{g}^2}\,\sqrt{2}\,[\sigma(x_3=\infty) - \sigma(x_3=-\infty)]\right| = \left|\frac{4\pi}{\tilde{g}^2}\,\left(m_{j_0} - m_{j_0+1}\right)\right|. \tag{7.8}$$

It is straightforward to check that this kink spectrum coincides with the monopole spectrum of the dual bulk theory in the quasiclassical approximation.

7.3 Dual weighted CP(N_f-1) model

Let us start from Hanany and Tong. The brane-based arguments of [11, 14] can be applied to the dual bulk theory too. This leads us to a dual weighted CP(N_f-1). Now it has $\tilde{N}$ orientational moduli $\tilde{n}^w_j$ with the U(1) charge +1. In addition, it has N size moduli $\tilde{z}^w_i$ with the U(1) charge −1. The bosonic action of this model is

$$S^{\rm D}_{\rm HT} = \int d^2x\left\{|\nabla_k\tilde{n}^w_j|^2 + |\tilde{\nabla}_k\tilde{z}^w_i|^2 + \frac{1}{4\tilde{e}^2}F^2_{kl} + \frac{1}{\tilde{e}^2}\,|\partial_k\sigma|^2\right.$$
$$\left. + \left|\sqrt{2}\sigma + m_j\right|^2\left|\tilde{n}^w_j\right|^2 + \left|\sqrt{2}\sigma + m_i\right|^2|\tilde{z}^w_i|^2 + \frac{\tilde{e}^2}{2}\left(|\tilde{n}^w_j|^2 - |\tilde{z}^w_i|^2 - \frac{4\pi}{\tilde{g}^2}\right)^2\right\},$$
$$i = 1, ..., N, \qquad j = 1, ..., \tilde{N}\,. \tag{7.9}$$

It is easy to see that the classical vacua of this model are at

$$\sqrt{2}\sigma = -m_{j_0}, \qquad \tilde{n}^w_{j_0} = \sqrt{\frac{4\pi}{\tilde{g}^2}}, \qquad \tilde{n}^w_{j\neq j_0} = 0, \qquad \tilde{z}^w_i = 0. \tag{7.10}$$

The quasiclassical spectrum of the dual weighted CP($N_f - 1$) model (7.9) can be obtained along the same lines as in Sec. 6.2. It appears to be the same as in the dual *zn* theory (7.4), see (7.6) and (7.8).

In passing we should mention the following. It turns out that the weighted CP($N_f - 1$) model is selfdual [34, 46, 47]. At $\xi \gg \Lambda^2$ the original theory is at weak coupling, and (2.13) is positive. Analytically continuing to the domain $\xi \ll \Lambda^2$, we formally make it negative, which signals, of course, that the low-energy description in terms of the original model is inappropriate. At $\xi \ll \Lambda^2$ the coupling of the infrared free dual bulk theory is given by

$$\frac{8\pi^2}{\tilde{g}^2}(\xi) = (N - \tilde{N}) \ln \frac{\Lambda}{\sqrt{\xi}} = -\frac{8\pi^2}{g^2}(\xi)\,. \tag{7.11}$$

It becomes positive and the dual model assumes the role of the legitimate low-energy description (at weak coupling). A direct inspection of the dual theory action (7.9) shows that the dual theory can be interpreted as a continuation of the sigma model (6.1) to negative values of the coupling constant g^2, where we identify

$$\tilde{n}^w_j = z^w_j, \qquad \tilde{z}^w_i = n^w_i, \qquad i = 1, ..., N, \qquad j = 1, ..., \tilde{N}. \tag{7.12}$$

8 Conclusions

Our task was to work out an honest-to-god field-theoretic derivation of the world-sheet theory for non-Abelian semilocal strings. The goal is achieved. The occurrence of the large IR parameter (1.3) not seen in the D-brane derivation proved to be crucial. In the limit when IR logarithm is large the world-sheet theory is obtained exactly. On the string world sheet we discovered a so far unknown $\mathcal{N} = 2$ two-dimensional sigma model, the *zn* model, with or without twisted masses. Alternative formulations of the *zn* model are worked out: conventional and extended gauged formulations and a geometric formulation. We compare the exact metric of the *zn* model with that of the weighted CP($N_f - 1$) model conjectured by Hanany and Tong, through D-branes. In fact these two models are essentially different. This

has been unequivocally demonstrated in certain regimes. Still quasiclassical excitation spectra of two models coincide. An obvious task for the future is the large-N solution of the zn model.

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A Appendix

A1. Useful formulae

For convenience we list here all the relevant traces which appear in the kinetic term for matter fields (2.36).

$$
\begin{aligned}
&\partial_k n^* \partial_k n + (\partial_k n^* n)^2 \equiv \left[\mathbb{C}P^{N-1}\right] , \\
&\mathrm{Tr}\left\{ \left[\partial_k (n\, n^*)\right] \cdot \left[\partial_k (n\, n^*)\right] \right\} = 2\left[\mathbb{C}P^{N-1}\right] , \\
&\mathrm{Tr}\left\{ \left[n\, n^*\right] \cdot \left[\partial_k n\, n^* - n\, \partial_k n^* - 2n\, n^* (n^* \partial_k n)\right] \cdot \left[\partial_k (n\, n^*)\right] \right\} = -\left[\mathbb{C}P^{N-1}\right] , \\
&\mathrm{Tr}\left\{ \left[\partial_k n\, n^* - n\, \partial_k n^* - 2n\, n^* (n^* \partial_k n)\right] \cdot \left[\partial_k (n\, n^*)\right] \right\} = 0 , \\
&\mathrm{Tr}\left\{ \left[n^*\right] \cdot \left[\partial_k n\, n^* - n\, \partial_k n^* - 2n\, n^* (n^* \partial_k n)\right] \cdot \left[\partial_k n\right] \right\} = -\left[\mathbb{C}P^{N-1}\right] , \\
&\mathrm{Tr}\left\{ \left[\partial_k n\, n^* - n\, \partial_k n^* - 2n\, n^* (n^* \partial_k n)\right]^2 \right\} = -2\left[\mathbb{C}P^{N-1}\right] , \\
&\mathrm{Tr}\left\{ \left[n\, n^*\right] \cdot \left[\partial_k n\, n^* - n\, \partial_k n^* - 2n\, n^* (n^* \partial_k n)\right]^2 \right\} = -\left[\mathbb{C}P^{N-1}\right] .
\end{aligned}
\tag{A.1}
$$

A2. Matter fields contributions

The matter field contribution is evaluated and decomposed in terms of the dependence on powers of the profile functions ω and γ

$$
\mathrm{Tr}\left[(\nabla_k Q)^* (\nabla_k Q)\right] \;=\; \mathcal{L}_{\omega^0 \gamma^0} + \mathcal{L}_{\omega^1} + \mathcal{L}_{\omega^2} + \mathcal{L}_{\gamma^1} + \mathcal{L}_{\gamma^2} \,, \tag{A.2}
$$

where

$$
\begin{aligned}
\mathcal{L}_{\omega^0\gamma^0} &= \mathrm{Tr}\Bigg\{\Big[\partial_k(\phi_1 - n\,n^*(\phi_1-\phi_2))\Big]\Big[\partial_k(\phi_1 - n\,n^*(\phi_1-\phi_2))\Big]\Bigg\} \\
&+ \Big[\partial_k(n^*\phi_3^*)\Big]\Big[\partial_k(n\,\phi_3)\Big] \\
&= 2(\phi_1-\phi_2)^2\Bigg((\partial_k n^*\partial_k n) + (\partial_k n^* n)^2\Bigg) + \partial_k(n^*\phi_3^*)\partial_k(n\,\phi_3) \\
&+ |\partial_k\phi_1|^2 + |\partial_k\phi_2|^2 \\
&= 2(\phi_1-\phi_2)^2\Big[\mathbb{C}P^{N-1}\Big] + \partial_k(n^*\phi_3^*)\partial_k(n\,\phi_3) + |\partial_k\phi_1|^2 + |\partial_k\phi_2|^2\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\omega^1} &= \mathrm{Tr}\Bigg\{\Big[\phi_1 - n\,n^*(\phi_1-\phi_2)\Big]\Big[\partial_k n\,n^* - n\,\partial_k n^* - 2n\,n^*(n^*\cdot\partial_k n)\Big] \\
&\times \Big[\partial_k(\phi_1 - n\,n^*(\phi_1-\phi_2))\Big]\Bigg\}\,\omega \\
&+ \Big[n^*\phi_3^*\Big]\Big[\partial_k n\,n^* - n\,\partial_k n^* - 2n\,n^*(n^*\partial_k n)\Big]\Big[\partial_k(n\,\phi_3)\Big]\omega + \mathrm{c.c.} \\
&= -2\Bigg((\phi_1-\phi_2)^2 + |\phi_3|^2\Bigg)\omega\Big[\mathbb{C}P^{N-1}\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\omega^2} &= -\mathrm{Tr}\Bigg\{\Big[\phi_1 - n\,n^*(\phi_1-\phi_2)\Big]\Big[\partial_k n\,n^* - n\,\partial_k n^* - 2n\,n^*(n^*\partial_k n)\Big]^2 \\
&\times \Big[\phi_1 - n\,n^*(\phi_1-\phi_2)\Big]\Bigg\}\omega^2 \\
&- \Big[n^*\,\phi_3^*\Big]\Big[\partial_k n\,n^* - n\,\partial_k n^* - 2n\,n^*(n^*\partial_k n)\Big]^2\Big[n\,\phi_3\Big]\omega^2 \\
&= \Bigg(2\phi_1^2 - 2(\phi_1-\phi_2)\phi_2 + (\phi_1-\phi_2)^2 + |\phi_3|^2\Bigg)\omega^2\Big[\mathbb{C}P^{N-1}\Big] \\
&= \Bigg(\phi_1^2 + \phi_2^2 + |\phi_3|^2\Bigg)\omega^2\Big[\mathbb{C}P^{N-1}\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\gamma^1} &= \mathrm{Tr}\Big\{\big[\phi_1 - n\,n^*(\phi_1-\phi_2)\big]\big[n\,n^*\big] \\
&\times \big[\partial_k(\phi_1 - n\,n^*(\phi_1-\phi_2))\big]\Big\}(\rho^*\partial_k\rho - \rho\partial_k\rho^* + 2|\rho|^2\,n^*\partial_k n)\gamma \\
&+ \big[n^*\phi_3^*\big]\big[\partial_k(n\,\phi_3)\big](\rho^*\partial_k\rho - \rho\partial_k\rho^* + 2|\rho|^2\,n^*\partial_k n)\gamma + \text{c.c.} \\
&= (\phi_3^*\partial_k\phi_3 - \phi_3\partial_k\phi_3^* + 2|\phi_3|^2 n^*\partial_k n)(\rho^*\partial_k\rho - \rho\partial_k\rho^* + 2|\rho|^2\,n^*\partial_k n)\gamma\,, \\
\mathcal{L}_{\gamma^2} &= -\mathrm{Tr}\Big\{\big[\phi_1 - n\,n^*(\phi_1-\phi_2)\big]\big[n\,n^*\big]\big[n\,n^*\big] \\
&\times \big[\phi_1 - n\,n^*(\phi_1-\phi_2)\big]\Big\}(\rho^*\partial_k\rho - \rho\partial_k\rho^* + 2|\rho|^2\,n^*\partial_k n)^2\gamma^2 \\
&- \big[n^*\,\phi_3^*\big]\big[n\,n^*\big]\big[n\,n^*\big]\big[n\,\phi_3\big](\rho^*\partial_k\rho - \rho\partial_k\rho^* + 2|\rho|^2\,n^*\partial_k n)^2\gamma^2 \\
&= -(\phi_2^2 + |\phi_3|^2)(\rho^*\partial_k\rho - \rho\partial_k\rho^* + 2|\rho|^2\,n^*\partial_k n)^2\gamma^2\,. \qquad \text{(A.3)}
\end{aligned}
$$

A3. Evaluation on the semilocal solution

In this section we explicitly evaluate the integrations over the transverse plane. We collect all terms appearing in (2.35) and (2.36) in terms of various combinations of derivatives,

$$
\begin{aligned}
&\mathcal{L}_{\partial_k n^*\partial_k n + (\partial_k n^* n)^2} = 2\pi\int r\,dr\Big\{\frac{1}{g^2}\left(\frac{2}{r^2}f^2(1-\omega)^2 + 2\omega'^2\right) \\
&+\Big[2\frac{\phi_2}{\sqrt{\xi}}(\sqrt{\xi}-\phi_2)^2 + \frac{|\rho|^2}{r^2}|\phi_2|^2(-1+2\frac{\phi_2}{\sqrt{\xi}}) + \left(\xi + |\phi_2^2|(1+\frac{|\rho|^2}{r^2})\right)\left(1-\frac{\phi_2}{\sqrt{\xi}}\right)^2\Big]\Big\} \\
&\times\big[\partial_k n^*\partial_k n + (\partial_k n^* n)^2\big] \\
&2\pi\int r\,dr\Big\{\xi\left(1-\frac{|\phi_2|^2}{\xi}\right)^2 + \frac{|\phi_2|^4}{\xi}\frac{|\rho|^2}{r^2}\Big\}\big[\partial_k n^*\partial_k n + (\partial_k n^* n)^2\big] \\
&= \frac{2\pi}{g^2}\big[\partial_k n^*\partial_k n + (\partial_k n^* n)^2\big] + 2\pi\xi\ln\frac{L}{|\rho|}|\rho|^2\big[\partial_k n^*\partial_k n + (\partial_k n^* n)^2\big]\,. \qquad \text{(A.4)}
\end{aligned}
$$

Note that the $1/g^2$ corrections drops out from the second piece. Furthermore,

$$
\begin{aligned}
&\mathcal{L}_{[\rho^*\partial_k\rho-\rho\partial_k\rho^*+2|\rho|^2(n^*\partial_k n)]^2} \\
&= 2\pi\int rdr\Big\{-\frac{1}{g^2}\left(\gamma'^2\right)+\frac{1}{r^2}|\phi_2|^2\gamma-|\phi_2^2|(1+\frac{|\rho|^2}{r^2})\gamma^2\Big\} \\
&\times[\rho^*\partial_k\rho-\rho\partial_k\rho^*+2|\rho|^2\,n^*\partial_k n]^2 \\
&= 2\pi\int rdr\Big\{-\frac{1}{g^2}\frac{r^2}{(r^2+|\rho|^2)^4}+\frac{1}{4r^2(r^2+|\rho|^2)}|\phi_2|^2\Big\} \\
&\times[\rho^*\partial_k\rho-\rho\partial_k\rho^*+2|\rho|^2\,n^*\partial_k n]^2 \\
&= \Big\{-\frac{\pi}{6g^2}\frac{1}{|\rho|^4}+\frac{\pi\xi}{4}\frac{1}{|\rho|^2}-\frac{\pi}{3g^2}\frac{1}{|\rho|^4}\Big\} \\
&\times[\rho^*\partial_k\rho-\rho\partial_k\rho^*+2|\rho|^2\,n^*\partial_k n]^2\,, \qquad (A.5)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{L}_{[\partial_k|\rho|^2]^2} \\
&= 2\pi\int rdr\Big\{\frac{1}{g^2}\frac{1}{r^2}(\partial_{|\rho|^2}f)^2+\Big[\Big(1+\frac{|\rho|^2}{r^2}\Big)(\partial_{|\rho|^2}\phi_2)^2+\frac{1}{r^2}\phi_2\,\partial_{|\rho|^2}\phi_2\Big]\Big\}\left[\partial_k|\rho|^2\right]^2 \\
&= \Big\{\frac{\pi}{6g^2}\frac{1}{|\rho|^4}-\frac{\pi\xi}{4}\frac{1}{|\rho|^2}+\frac{\pi}{3g^2}\frac{1}{|\rho|^4}\Big\}\left[\partial_k|\rho|^2\right]^2, \qquad (A.6)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}_{|\partial_k\rho+\rho(n^*\partial_k n)|^2} &= 2\pi\int rdr\frac{1}{r^2}|\phi_2|^2|\partial_k\rho+\rho(n^*\partial_k n)|^2 \\
&= \Big\{2\pi\xi\ln\frac{L}{|\rho|}-\frac{2\pi}{g^2}\frac{1}{|\rho|^2}\Big\}|\partial_k\rho+\rho(n^*\partial_k n)|^2 \qquad (A.7)
\end{aligned}
$$

Collecting together all the pieces we obtain the result reported in (2.40), namely,

$$\begin{aligned}\mathcal{L}_{\rm eff} &= \pi\xi \ln \frac{L^2}{|\rho|^2}\big|\partial_k(\rho\, n)\big|^2 - \pi\xi|\partial_k\rho + \rho\,(n^*\partial_k n)|^2 \\ &+ \frac{2\pi}{g^2}\left[\partial_k n^*\partial_k n + (\partial_k n^* n)^2\right]\end{aligned} \tag{A.8}$$

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